Visual Mathematics in Practice

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Theme of the lesson:	Trigonometry
Place in curriculum: (type of school, grade)	Gymnasium, 2
Age of the students/pupils:	16-17
Title of the lesson:	The application of trigonometry

Description of the lesson

Time	Exercises, matters, parts of the lesson	Methods and forms of student activities	Developable competencies
Please, write cc. the minutes you spent with each activities	**1 The application of trigonometry** **1.1 Sinus theorem** We will observe a triangle that is not rectangular. In order to solve this triangle you need to know at least three of its elements, one of which must be a side. We distinguish the following cases (marked in red are given elements of the triangle): ▪ one side and any two angles (ASA, AAS),	*individual work, work in groups, .*	*the thinking, learning, communication skills, visualization*

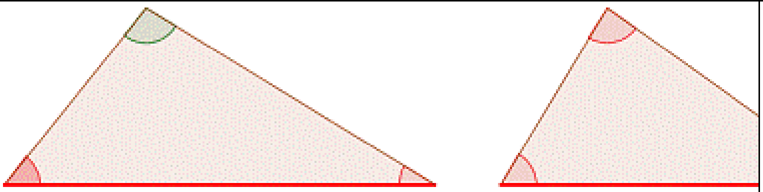


Image 1.1.

- two sides and an angle opposite one of them (SSA),

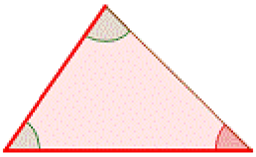


Image 1.2.

- two sides and the angle of the affected (SAS),

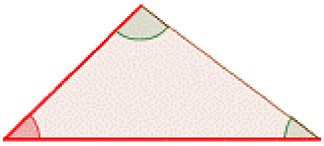


Image 1.3.

- three sites (SSS).

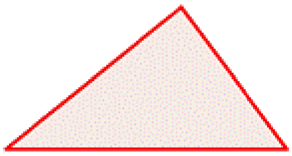


Image 1.4.

Theorem. Sinus theorem

For every triangle the following term is true

$$\frac{\sin \alpha}{a} = \frac{\sin \beta}{b} = \frac{\sin \gamma}{c}.$$

Proof. Consider first a blunt triangle ABC where the blunt angle is in the point C. Let us then extend the line segment AC, followed by projecting the point B to that extension so that they are perpendicular, and mark it D - image 1.5.

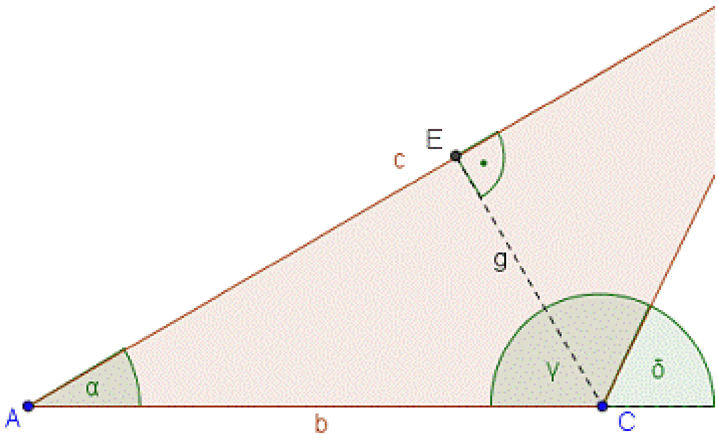


Image1.5.

Now the angles α and β are angles of right-angled triangle and then we know that

$$\sin\alpha = \frac{g}{b}, \quad g = b \cdot \sin\alpha$$

and

$$\sin\beta = \frac{g}{a}, \quad g = a \cdot \sin\beta$$

From this directly follows

$$b \cdot \sin\alpha = a \cdot \sin\beta$$

respectively

$$\frac{\sin\alpha}{a} = \frac{\sin\beta}{b}.$$

Using the large right-angled triangle from the image 1.5 we get

$$\sin\alpha = \frac{h}{b},$$

Followed by

$$h = b \cdot \sin\alpha$$

and

$$\sin\delta = \frac{h}{a},$$

respectively

$$h = a \cdot \sin\delta$$

As the

$$\delta = \pi - \gamma$$

and

$$\sin\delta = \sin(\pi - \gamma) = \sin\gamma$$

we get

$$c \cdot \sin\alpha = a \cdot \sin\delta = a \cdot \sin\gamma .$$

From this directly follows

$$c \cdot \sin\alpha = a \cdot \sin\gamma$$

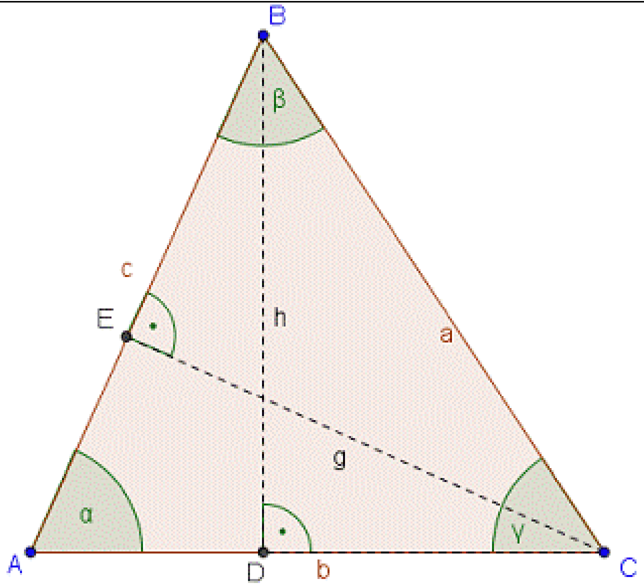
respectively

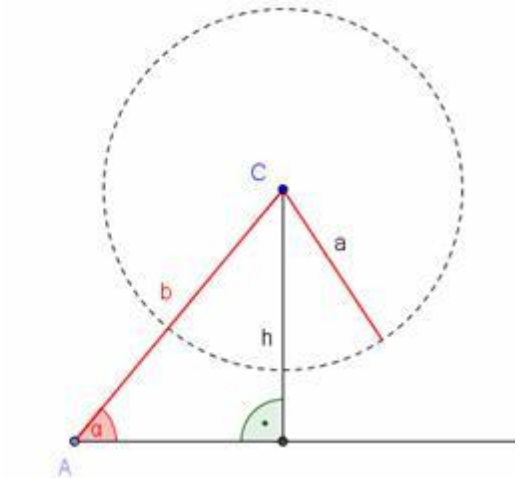
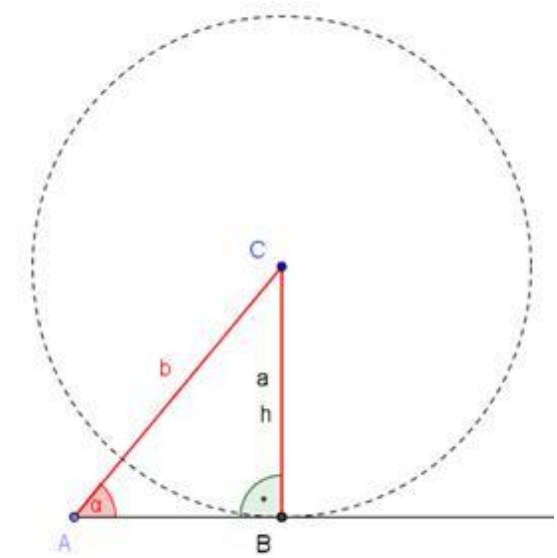
$$\frac{\sin\alpha}{a} = \frac{\sin\gamma}{c} .$$

In the case where the angle is an acute-angled proof is completely analogous. Then point D belongs to the site b.

That concludes the proof of the sine theorem.

■Using 1.5.ggb we will consider the evidence for both cases of the proof. Move point B to cross from one situation to another and demonstrate the same character evidence in both cases.

	 Image 1.6. A particularly interesting case when applying the sine theorem is when the two sides are known and the angle opposite one of them (SSA). We will consider this case in more detail. Lets take the given sites a and b and the angle α. Ffirst let us assume that the angle α is acute, $0^\circ < \alpha < 90^\circ$. Draw the angle α in a standard position and mark the point C on the moving branch so that the length of the longer line segment defined by the point A of the angle α and the point C is equal to the length of the side b. Let h be the distance of the point C from the stationary branch angle α. Depending on the side a we have the following situation ● If $a < h$, then there is no a triangle with the given elements, image 1.7.		

	 Слика 1.7. $a < h$ no triangle ● If $a = h$, then there is only one right-angled triangle with the given elements, image 1.8.  Image1.8. $a = h$ one triangle ● If $h < a < b$, then there are two triangles with the given elements, image 1.9.		

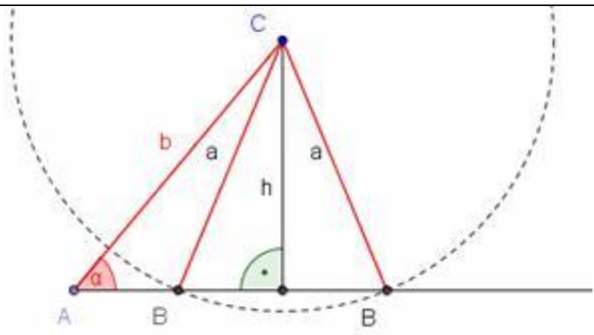


Image 1.9. $h < a < b$ two triangles

- If $a \geq b$, there is only one triangle with a given element.

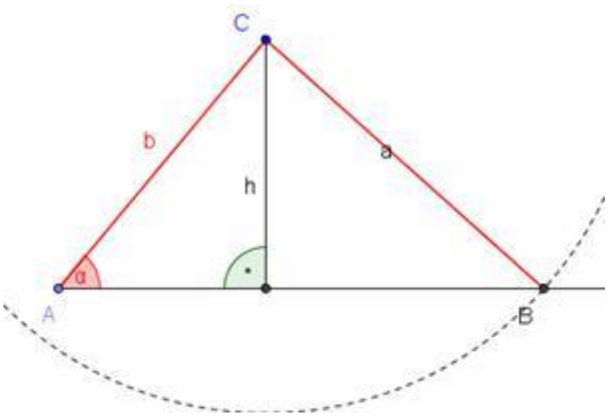


Image 1.10. $a \geq b$ one triangle

- If α is the obtuse angle, $90^\circ < \alpha < 180^\circ$ there are only two possibilities, as shown in images 1.11 and 1:12.

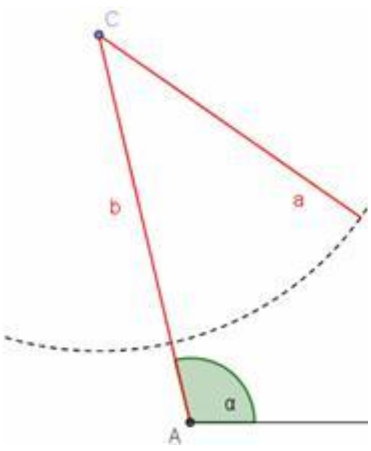


Image 1.11. $a \leq b$ no triangle

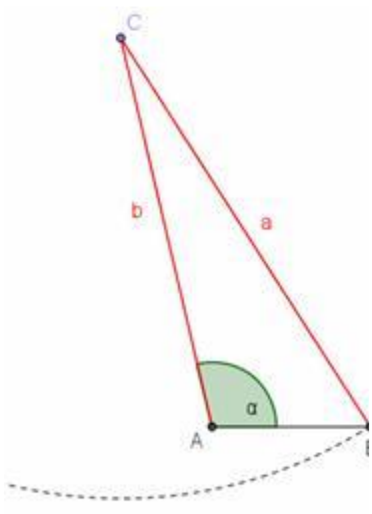


Image 5.12. $a > b$ one triangle

■ Using [1.7.ggb]() and [1.11.ggb](), observe the situation when the angle α is acute, $0^\circ < \alpha < 90^\circ$ and when the angle α is obtuse $90^\circ < \alpha < 180^\circ$. Change the length of the side a and size of the angle α.

1.2 Cosine theorem

When the triangle is given with two corners and a single side or when its given with two sides and an angle opposite one of them, then the other elements of the triangle can be determined by using the sine rule. However, when the triangle is given with two sides and the affected angle or given with the three sites, then we cannot determine the other elements of the triangle by using the sine theorem. In these cases we determine them by using the cosine theorem. This theorem is a generalization of the Pythagorean theorem and is applicable to every triangle.

Theorem. Cosine theorem

If the triangle ABC has sides a, b и c and angles α, β and γ, than

$$a^2 = b^2 + c^2 - 2bc\cos\alpha,$$

$$b^2 = a^2 + c^2 - 2ac\cos\beta,$$

$$c^2 = a^2 + b^2 - 2bc\cos\gamma.$$

Proof. Suppose that the position of the triangle is as shown in image 1.13 or as in image 1.14.

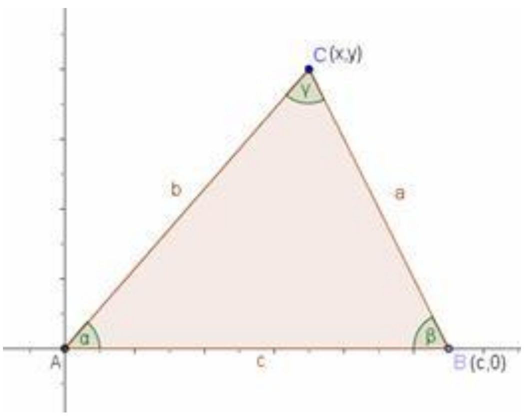


Image 1.13. $0^\circ < \alpha < 90^\circ$

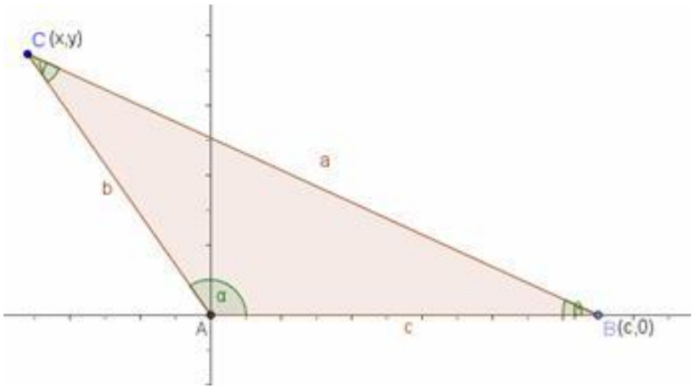


Image 1.14. $90^\circ < \alpha < 180^\circ$

In both cases the

$$x = b \cdot \cos\alpha$$

and

$$y = b \cdot \sin\alpha.$$

The length of the side BC is a. By calculating the distance between the two points we get

$$a = \sqrt{(x-c)^2 + (y-0)^2}$$

$$a = \sqrt{(b\cdot\cos\alpha - c)^2 + (b\cdot\sin\alpha)^2}$$

$$a^2 = (b\cdot\cos\alpha - c)^2 + (b\cdot\sin\alpha)^2$$

$$a^2 = b^2\cdot\cos^2\alpha + c^2 - 2bc\cdot\cos\alpha + b^2\cdot\sin^2\alpha$$

$$a^2 = b^2\left(\cos^2\alpha + \sin^2\alpha\right) + c^2 - 2bc\cdot\cos\alpha$$

$$a^2 = b^2\left(\cos^2\alpha + \sin^2\alpha\right) + c^2 - 2bc\cdot\cos\alpha$$

As the

$$\cos^2\alpha + \sin^2\alpha = 1,$$

we get first assertion theorem.

In the same way, by watching the triangle where the point B has coordinates $(0,0)$, and the point C has coordinates $(0,0)$, we obtain proof of the other two cases.

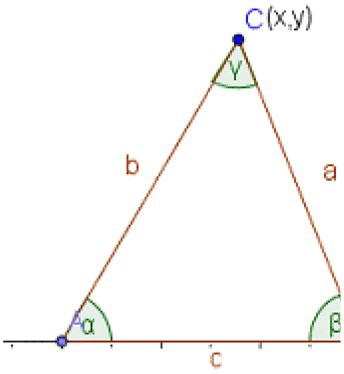


Image 1.15.

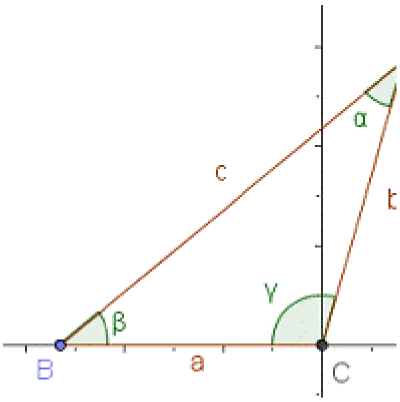


Image 1.16.

1.3 Surface Area of the triangle

It is known that the surface area of the triangle can be calculated by using one side and its corresponding triangle height. If a is the side and h is the appropriate height, then the area of the triangle is

$$P = \frac{1}{2}ah.$$

Consider the triangle shown in image 1.17 where the angle γ is acute. As

$$\sin\gamma = \frac{h}{b}$$

we get

$$h = b \sin \gamma .$$

Substituting h with $b \sin \alpha$ we get

$$P = \frac{1}{2} ab \sin \gamma .$$

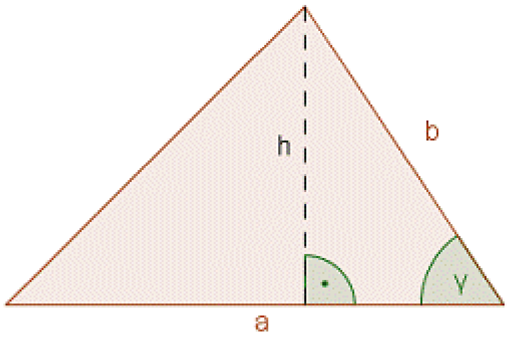


Image 1.17.

The resulting configuration is correct even for the other situations, when the angle γ is perpendicular or obtuse. The surface area of a triangle can be calculated in the same manner if instead of the angle γ we known the angle α or the angle β. Thus we obtain the following theorem.

Theorem. Surface area of the triangle

The surface area of a triangle is equal to half the product of two sites of a triangle multiplied by the sine of the affected angle.

$$P = \frac{1}{2} ab \sin \gamma ,$$

$$P = \frac{1}{2} ac \sin \beta ,$$

$$P = \frac{1}{2} bc \sin \alpha$$

Example 1

Calculate the area of a triangle shown in image 1.18, if the sides are given in centimeters.

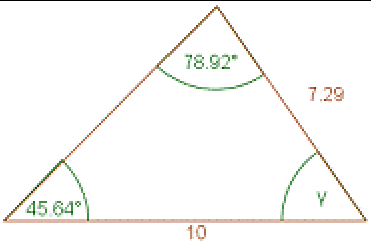


Image 1.18.

Solution.

We obtain the angle γ from $\alpha + \beta + \gamma = 180^\circ$:

$\gamma = 180^\circ - 78.92^\circ - 45.64 = 55.44^\circ$.

Using the formula of the previous theorem we get

$$P = \frac{1}{2} \cdot 10 \cdot 7.29 \cdot \sin 55.44^\circ \approx 30$$.

So, the area of the triangle is approximately $30\,cm^2$.

Examle 2.

Calculate the surface area of the quadrilateral in the image 1.19 if the sides are given in centimetres.

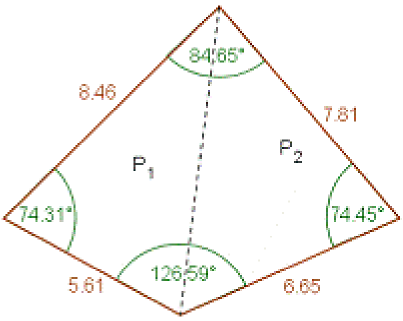


Image 1.19.

Solution.

If you divide the quadrilateral into two triangles as shown in image 1:19, then we can calculate the area of each section

$$P_1 = \frac{1}{2} \cdot 5.61 \cdot 8.46 \cdot \sin 74.31^\circ \approx 22.85$$

$$P_2 = \frac{1}{2} \cdot 6.65 \cdot 7.81 \cdot \sin 74.45^{\circ} \approx 25.02,$$

The surface area of the quadrilateral is $P_1 + P_2 \approx 47.87\ cm^2$.

1.4 Repetition

Sinus theorem

- To solve a triangle we are required to know at least three of its elements, out of which one must be the length of one of its sides.
- We solve the situations ASA and SSA by applying the sine theorem.
- The Sine theorem states that the ratio of the sine of one angle and its appropriate side is the same for all the angles and their appropriate sides.

  ($\frac{\sin \alpha}{a} = \frac{\sin \beta}{b} = \frac{\sin \gamma}{c}$).

- For the situation SSA the length of the sides and the size of the angle influance the number of triangles that can be constructed two, one or none.

Cosines theorem

- We solve the situations SAS and SSS by applying the cosine theorem.
- The cosine theorem states that the square of one side of the triangle is equal to the sum of the squares of the other two sides lessend by double the value of the product of those sides and the cosine of the angle between them.:

$$c^2 = a^2 + b^2 - 2bc \cos \gamma, \quad a^2 = b^2 + c^2 - 2bc \cos \alpha,$$
$$b^2 = a^2 + c^2 - 2ac \cos \beta.$$

1.5 Задаци

Determen the number of triangles that can be constructed with the given elements

1. $a = 5,\ b = 3,\ c = 8$

2. $a = 5$, $b = 14$, $c = 8$

3. $a = 5$, $b = 10$, $c = 8$

4. $a = 15$, $b = 3$, $c = 18$

Solve the following triangles with the given elements.

5. $a = 5$, $b = 3$, $\gamma = 60^\circ$

6. $a = 5$, $c = 3.5$, $\beta = 10^\circ$

7. $a = 15.2$, $b = 13.3$, $c = 8.4$

8. $b = 3.5$, $a = c.2$, $\alpha = 60^\circ$

9. $a = 5.6$, $b = 3.9$, $\alpha = 60^\circ$

10. $a = 5$, $\beta = 36^\circ$, $\gamma = 64^\circ$

11. $b = 3$, $\alpha = 40^\circ$, $\gamma = 120^\circ$,

Calculate the surface area of the following triangle if the sides are given in meters.

12. $a = 8.3$, $b = 8.4$, $\gamma = 46^\circ$

13. $a = 8$, $c = 6.4$, $\gamma = 132^\circ$

14. Analise the situation SSA by using the applications 1.7.ggb и 1.11.ggb.

The inspiration came from lecture Djurdjica Takaci

Summary

The autonomy of students in the class activities, the ability to independently observed relations. Visualization facilitate and accelerate the adoption of mathematical concepts.

Supplements

Used materials:	
Photos:	*If you have made photos about the lesson or the products of the lesson, please add some (also you can send it in another file, and just mention the name of the file here)*