Inspired by Escher

Tessellation is a mathematical process of creating a plane by using only congruent geometrical shapes. This project is based on inter-dependence of mathematics and art. Escher, the most famous artist who used tessellation in his works, presented numerous solutions to this problem. For this project, Escher's studies are used together with analysis and suggestions of the author of this paper. We are relying on theorems and proofs which are well-known in the world of mathematics for solving the problems of tessellation. We will show that this problem is a problem of a modern world, frequently addressed and especially used in the area of graphical design. The point of tessellation lies in philosophical understanding of the world through mathematical terms which were universally used in language of artists ever since the ancient period. Remarkable achievements can be seen in the culture of Islam as well as in the works of modern designers who use technology in order to program new shapes of infinity in virtual worlds.

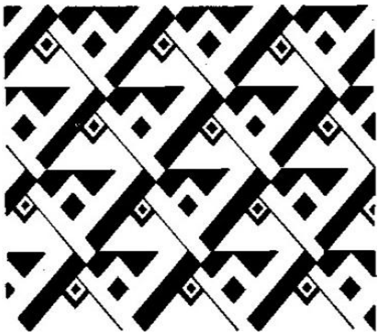
Looking for patterns

This is how Richard Feynman, an American physicist and Nobel Laureate, describes his first mathematical experiences at the hand of his father:

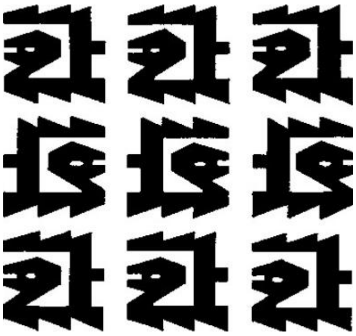
…when I still ate in a high chair; my father would play a game with me after dinner. He had bought a whole lot of odd rectangular bathroom floor tiles from someplace in Long Island City. We set them up on end one next to the other, and I was allowed to push the end one and watch the whole thing go down. So far so good.
Next the game improved. The tiles were different colors. I must put one white, two blues, one white, two blues and another white and then two blues… You recognize already the usual insidious cleverness; first delight him in play, and then slowly inject material of educational value! Well, my mother, who is a much more feeling woman, began to realize the insidiousness of his efforts and said, "Mel, please let the poor child put a blue tile if he wants to." My father said, "No, I want him to pay attention to patterns. It is the only thing I can do that is mathematics at this earliest level." If I were giving a talk on "what is mathematics?" I would have already answered you. Mathematics is looking for patterns. [4]

But, are the mathematicians unique in their search for patterns? Or is it in our nature, especially when we talk about geometrical shapes and tiling? If we look around, we shall see that we are surrounded by tiles and patterns – we tile bathroom and kitchen walls, pavements, floors; we play jigsaw puzzles throughout our childhood. Even nature itself has its own patterns and tiling – pineapple, honeycomb… and they all do that by following one simple rule – gaps and overlaps are not allowed. We do it unconsciously too. However, there are certain people who are very much aware of what they do - mathematicians on one hand and decorators and artists on the other. Their work is interlaced. The first ones explore and search for the rules and patterns, the other ones use those to adorn our environment. And there is a place for all of them. As Dutch artist Maurits Cornelis Escher wrote - mathematicians gave us a definition of the concept, they determined and counted the ways of dividing the plane in a regular way. Having done that, they opened the door of an enormous area, but they did not enter it. Naturally, they were much more interested in finding out the way to open the door than in the garden behind it.

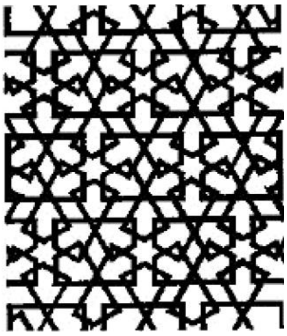
Throughout the centuries painters and decorators have been enriching our environment. The following are just some examples of their art.



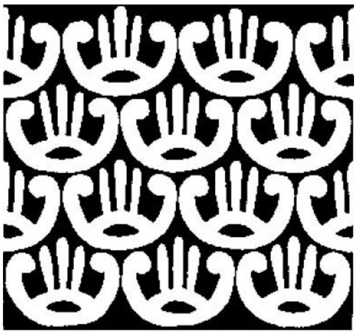
Pattern from the land of Incas



Peruvian drawing - birds



Rectangles and hexagons from Arabic peninsula



Phoenicians - design on a tomb in Syria

Japanese triangles

Escher himself was inspired by these ones

Alhambra - wall decoration

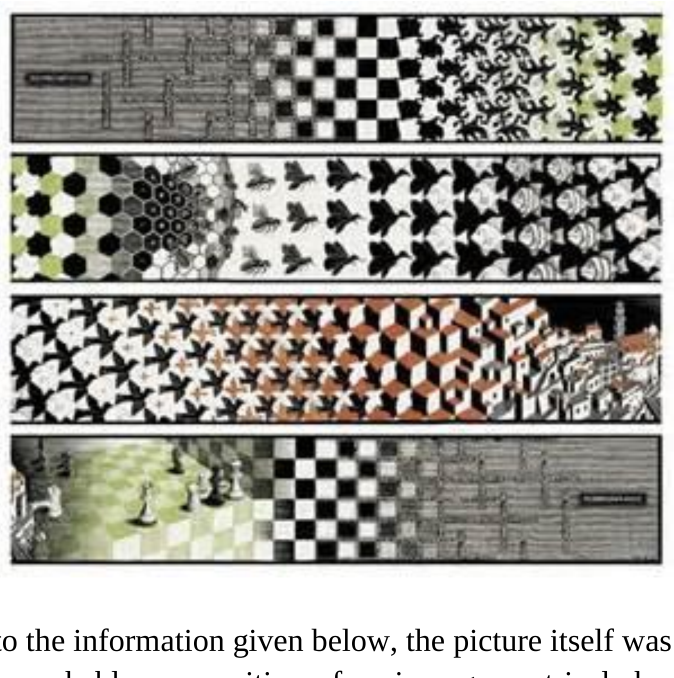
The door in Alhambra

And we were inspired by Escher.



Introduction

Having had a task to research graphics, we wanted to introduce something new and interesting to our classmates. In a long web search we came across a fascinating piece of graphic work. It was, as we found out immediately, "Metamorphosis II" by Dutch artist Maurits Cornelis Escher.



According to the information given below, the picture itself was four meters long, was an unusual and a remarkable composition of various geometrical shapes that slowly turned into living creatures and reversed again into previous or different shapes again and again. It was enough to make us even more interested in the further research.

Escher is famous for his prints, graphic works, lithographs and woodcuts. He researched on various fields – infinity, impossible constructions, tessellations and architectures. He found the inspiration for many of his famous works during the time he lived and travelled in Italy and Spain. Alhambra, the residence of the latest Moorish monarchs was the most interesting place for him. Escher was fascinated by versatility of the ancient artists in the same way we were delighted at seeing his graphics. He was so enchanted by regular tessellation, by patterns and ornaments that were so perfectly shaped following the rules of regular tessellation and symmetry that he introduced geometry into his works. He even became friends and worked with one of the most famous mathematicians of the time- Sir Richard Penrose, who also showed a lot of interest in the division of a plane topic.

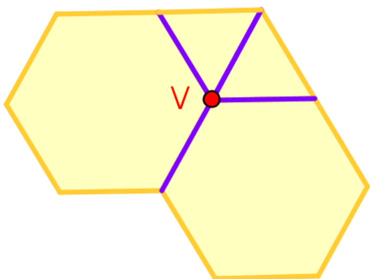
Extensive observation of Escher's print works led us to a conclusion that he used only three geometrical shapes, three regular polygons – triangle, square and hexagon as a starting point. Then he used to change their shape in such a way as to transform them into living creatures by following two simple rules – keeping the area and symmetry. The fact that he himself had researched the regular tessellation to improve his works brought us to the same goal. So, we set off to explore tessellation as well.

Math behind - what is tessellation?

Tessellation (or tiling) is the process of covering a surface with a pattern of flat shapes so that there are neither overlaps nor gaps. There are different kinds of tessellation – regular, semi-regular and demi-regular. We are going to explore regular and semi-regular ones, as did Escher. Regular tessellation is a pattern made by repeating only one regular polygon, while a semi-regular one is made of two or more regular polygons. We were surprised to see that the number of them was limited - there are only three regular, and eight semi-regular tessellations! That fact is a result of solving some Diophantine equations as we are going to prove.

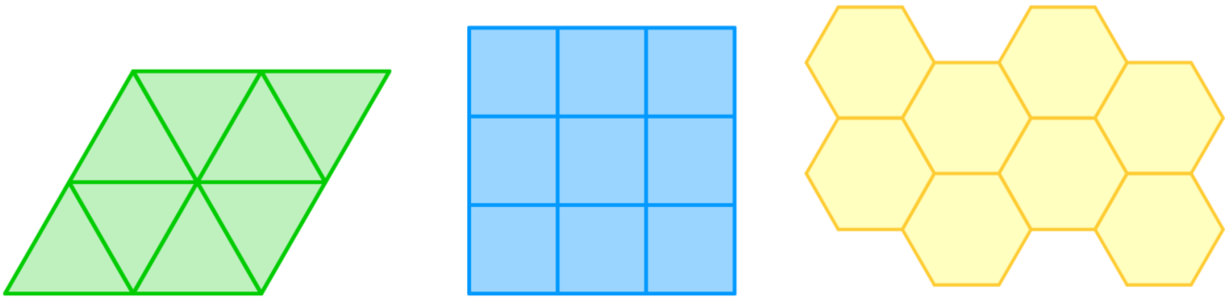
Regular tessellation

The point in which the polygons meet is called vertex – picture 1.



Picture 1

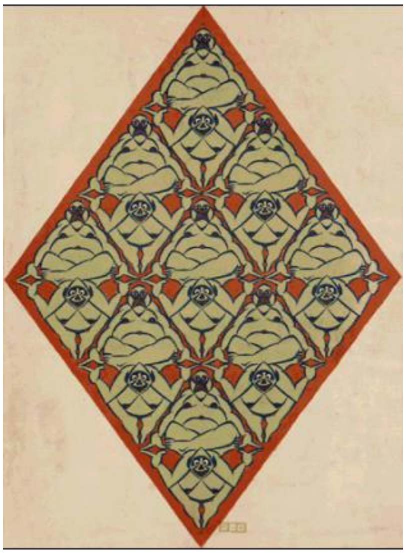
For regular and semi-regular tessellation, the pattern in each vertex must be the same. For regular tessellation that means that all angles of all polygons that meet in the same point are equal. So, if n is a number of sides of our regular polygon, then its angle will be $\alpha_n = \frac{(n-2)180°}{n}$. Let k be the number of regular polygons meeting in each vertex. Then, as the sum of all angles meeting in the same vertex is $360°$, and as there must not be overlaps, numbers n and k are the solutions of the following equation $k \cdot \alpha_n = 360°$, i.e. $k\frac{(n-2)\cancel{180°}}{n} = \cancel{360°}$. After rearranging it, the equation will look like this: $k = 2 + \frac{4}{n-2}$. Numbers k and n are positive integer numbers, so there are only few possibilities for *n-2*, i.e. $n-2 \in \{-4,-2,-1,1,2,4\}$. As $n \geq 3$, hence $(k,n) \in \{(6,3),(4,4),(3,6)\}$ - picture 2.



Picture 1

There are many examples of regular tessellation around us – honeycombs, chess boards, pavements, walls, and, of course, Escher's art.

Pavements in Novi Sad



M. C. Escher - Plane-filling Motif with Human Figures

Semi-regular tessellation

For semi-regular tessellation, the pattern of each vertex must be the same, but different regular polygons can be used. To prove that there are just eight patterns for semi-regular tessellation, we shall follow the similar process. First, as the sum of the angles of the first four regular polygons is $60°+90°+108°+120°=378°$, which is greater than 360^{o}, there cannot be more than three different regular polygons meeting in one vertex. We shall consider both cases.

I Semi regular tessellation with two regular polygons

Let k_1 and k_2 be the numbers of the polygons with n_1 and n_2 sides, respectively, meeting in the same vertex. Those integer numbers should satisfy the following inequalities

(1) $$3 \le k_1 + k_2 < 6$$

(2) $$n_1,\ n_2 \ge 3$$

and the similar Diophantine equation to the previous one, due to the sum of the angles

(3) $$k_1 \cdot \frac{n_1-2}{n_1} \cdot 180° + k_2 \cdot \frac{n_2-2}{n_2} \cdot 180° = 360°.$$

After simplifying the equation (3), it will transform to

(4) $$k_1\left(1-\frac{2}{n_1}\right) + k_2\left(1-\frac{2}{n_2}\right) = 2.$$

Being the integers and according to (1) the possibilities for k_1 and k_2 are given in the following table

k_1	1	1	2	1	2
k_2	2	3	2	4	3

Let's consider all the possibilities from the table, relative to equation (4):

1. Given values: $k_1=1$ and $k_2=2$ (or $k_1=2$, $k_2=1$)

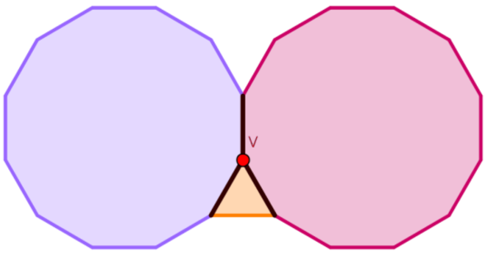
   By substituting given values in the equation (4), it will be $1-\frac{1}{n_1}+2\left(1-\frac{1}{n_2}\right)=2$, i.e. $n_2=4+\frac{8}{n_1-2}$. Therefore, possible values for n_1-2 are $\{-8,-4,-2,-1,1,2,4,8\}$.

   According to the inequality (2) solutions for n_1 are $\{3,4,6,10\}$, and then $n_2\in\{12,8,6,5\}$ respectively. Let's write these down in the table

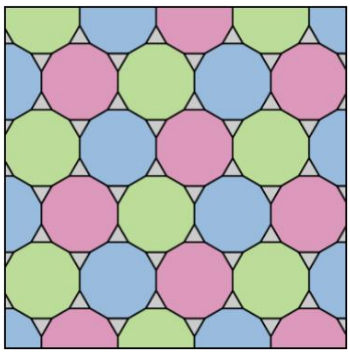
n_1	k_1	n_2	k_2
3	1	12	2
4	1	8	2
6	1	6	2
10	1	5	2

The third possibility is a regular tessellation (6, 6, 6), so there are three possibilities to discuss.

1.1. In each vertex a triangle and two polygons with twelve sides meet (Picture 3 and Picture 4.)

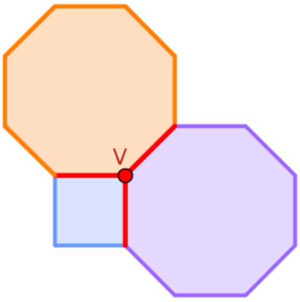


Picture 3 – the only one possible arrangement

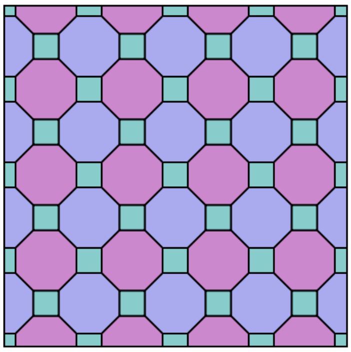


Picture 4

1.2. In each vertex a square and two octagons meet (Pictures 5, 6 and 7)



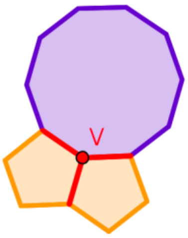
Picture 5



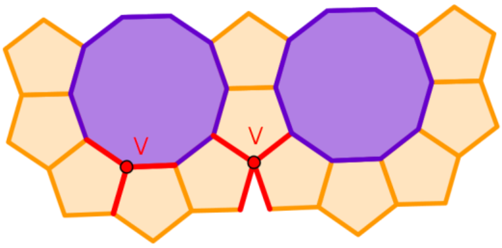
Picture 6

Picture 7 - Pavement in Novi Sad

1.3. In each vertex two pentagons and a polygon with ten sides meet. But this solution doesn't follow neither the rule that there are no gaps nor that the patterns in each vertex are equal (pictures 8 and 9)



Picture 8 – the only possible pattern

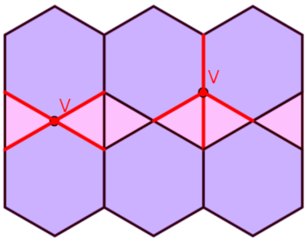


Picture 9 – pattern does not fit the rules

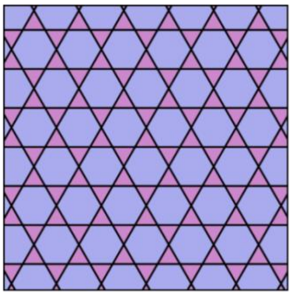
2. Given values: $k_1 = 1$ and $k_2 = 3$ (or $k_1 = 3$, $k_2 = 1$)

From the equation (4) and inequality (2) we get $n_2 = 3 + \dfrac{3}{n_1 - 1}$. Therefore, the only solution is $(n_1, n_2) = (4, 4)$ and that is a squared regular tessellation.

3. For $k_1 = k_2 = 2$ we get a Diophantine equation : $n_2 = 2 + \frac{4}{n_1 - 2}$, with only two possible solutions that fit the condition (2) and these are $(n_1, n_2) \in \{(3,6),(4,4)\}$. The solution (4, 4), as it was mentioned before is a regular tessellation, so in each vertex two triangles and two hexagons meet. There are two possible patterns (pictures 10 and 11), but in one of them the rule of the equality of the vertices is not satisfied (picture 10).

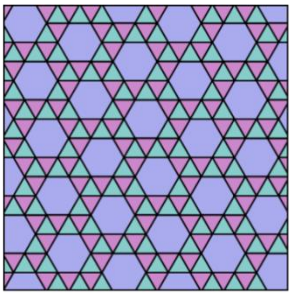


Picture 10 - Patterns in each vertex are not equal



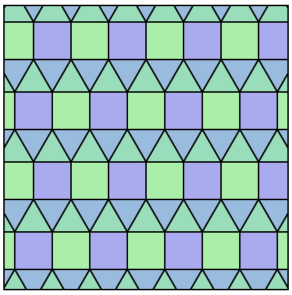
Picture 11 – Pattern that fit the rules of semiregular tesselation

4. For $k_1 = 1$ and $k_2 = 4$, the Diophantine equation $n_1 = \frac{2n_2}{3n_2 - 8}$ is to be solved. There are many possibilities for the denominator $3n_2 - 8 \in \{\pm1, \pm2, \pm n_2, \pm 2n_2\}$, but only one of them fit can satisfy all the conditions given, and it is a pair $(6,3)$. It means that a hexagon and four triangles meet in one vertex (Picture 12)

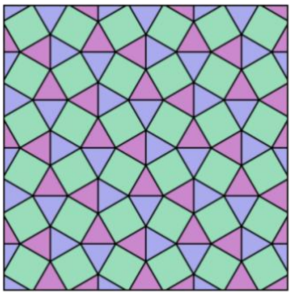


Picture 12

5. The last possibility is for $k_1 = 2$ and $k_2 = 3$. The Diophantine equation to solve is $n_1 = \frac{4n_2}{3n_2 - 6} = \frac{4n_2}{3(n_2 - 2)}$. According to (2) and to the previous equation, n_2 should be divisible by 3, so there can be only one solution and that is $n_2 = 3$ .Therefore, $n_1 = 4$. It means that three triangles and two squares meet in each vertex. There are two patterns that fit the rules – (3,3,3,4,4) in the picture 13, and (4,3,4,3,3) in the picture 14.



Picture 13



Picture 14

This is how Salvador Dali used one of them in his painting entitled ” 50 Abstract Paintings which Seen from Two Metres Change into Three Lenins Disguised as Chinese and Seen from Six Metres Appear as the Head of a Royal Tiger” (Picture 15)

Picture 15

II Semi regular tessellation with three regular polygons

Let k_1, k_2 and k_3 be the numbers of regular polygons with n_1 , n_2 and n_3 sides, respectively, meeting in the same vertex. Those numbers should fit the following relations due to the previous observations:

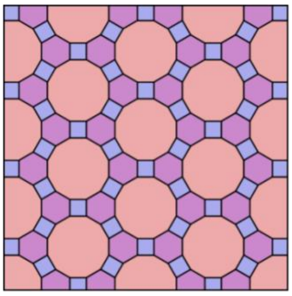
(5) $$k_1\left(1-\frac{2}{n_1}\right)+k_2\left(1-\frac{2}{n_2}\right)+k_3\left(1-\frac{2}{n_3}\right)=2$$ (due to the sum of angles in each vertex)

$$k_1,k_2,k_3 \in N$$
$$3 \le k_1+k_2+k_3 < 6$$
$$n_1,n_2,n_3 \ge 3.$$

The possibilities are given in the following table:

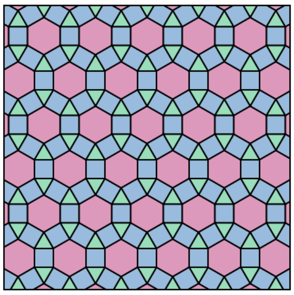
k_1	1	2	2	3
k_2	1	1	2	1
k_3	1	1	1	1

1. For $k_1=k_2=k_3=1$, Diophantine equation (5) is equivalent to the equation $\frac{1}{n_1}+\frac{1}{n_2}+\frac{1}{n_3}=\frac{1}{2}$, which has only one solution (4, 6, 12). That means that a square, a hexagon and a regular polygon of twelve sides meet in each vertex, as it is shown in the picture 16.



Picture 16

2. If $k_1 = 2$, $k_2 = k_3 = 1$, after transforming the equation (5), the equation to solve is $\frac{2}{n_1} + \frac{1}{n_2} + \frac{1}{n_3} = 1$ or $\frac{1}{n_1} + \frac{1}{2n_2} + \frac{1}{2n_3} = \frac{1}{2}$. The only possible solution is (4, 3, 6). It means that two squares, a triangle and a hexagon meet in each vertex and the only possible pattern for semi regular tessellation is given in the picture 17.



Picture 17

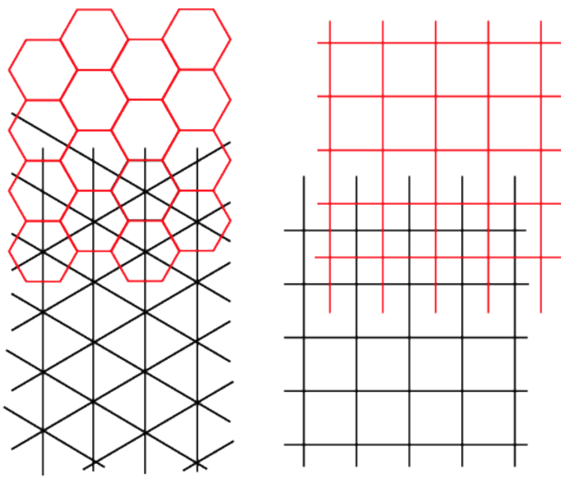
3. and 4. For both these cases there is no probable solution, due to the fact that the sum of the angles in each vertex is 360°. Namely, if we choose the triangles, squares and pentagons, which are the polygons with minimal number of sides, the sum will be greater than 360°.
   For $k_1 = k_2 = 2$ and $k_3 = 1$ it is $2 \cdot 60° + 2 \cdot 90° + 108° = 408°$
   For $k_1 = 3$ and $k_2 = k_3 = 1$ it is $3 \cdot 60° + 90° + 108° = 378°$
   In other cases, the sum is even greater.

We have exhausted all the possible regular and semi-regular tessellations.

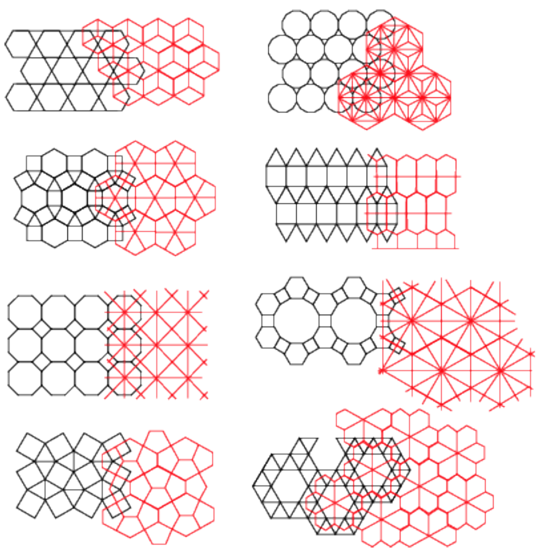
Other types of tessellation

Every type of regular and semi-regular tessellation has its dual one, which is formed if the centres of the polygons become vertices. So, triangular and hexagonal regular tessellations are dual to each other, and the squared one is dual to itself (picture 18).

The dual tessellations for semi regular ones are given in the picture 19. The last one in the first column is known as a Cairo tiling, as it is the most common tiling that is found on the streets of Cairo and in Islamic art in general.

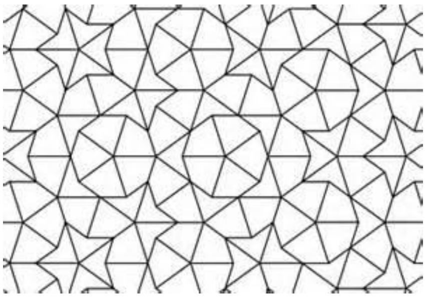


Picture 18

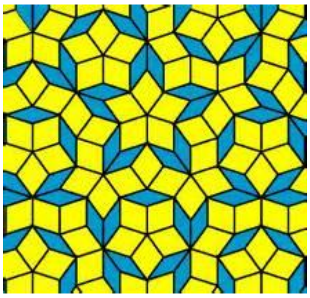


Picture 19

As previously mentioned**,** Escher worked on this subject with one of the most famous mathematicians, Sir Roger Penrose, who, after Escher had died, published his work of non-periodical tessellation, known as Penrose tiling (pictures 20 and 21).



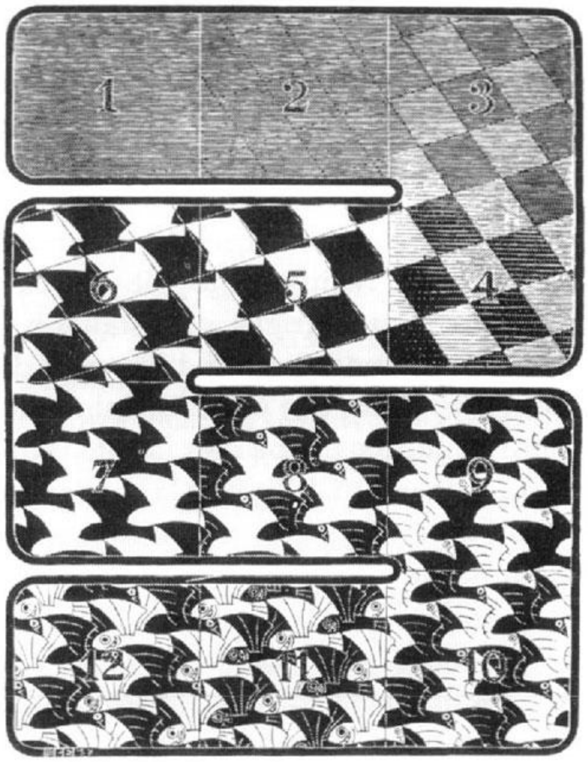
Picture 20



Picture 21

Escher and tessellation

Escher himself was fascinated by the Moorish wall decoration, whose authors had discovered and applied all the possible regular tessellations using all isometric transformations (translation, rotation and symmetry) in the XIII century. The fact that it was and still is forbidden to represent any kind of living creatures in Islam caused the application of isometrics reach its climax in Islamic art since it was the only way for the Muslims to express themselves artistically. But Escher made a step forward. Actually, he made a woodcut named “Regular division of a plane I” and in his book eng. “Researching infinity” *(Esplorando l'Infinito: i segreti di una ricerca artistic,* 1991) described his method in twelve steps (picture 22)



Picture 22 - Regular division of a plane I by M. C. Escher

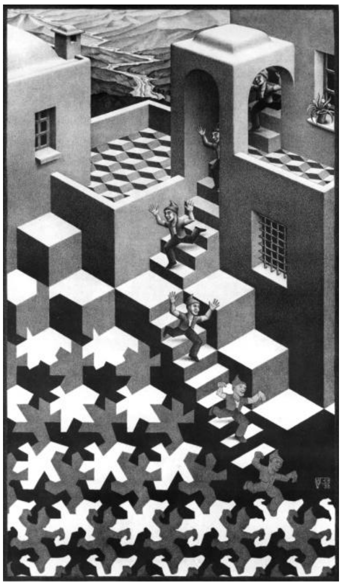
1. The beginning is grey – it is the way to show the stillness, the absence of time and space that precede life, but that return after one ends. Gray is the colour that represents the contrast between white and black that are generated by grey. This is to present the very beginning of development and action, so there is a question whether there was anything before it all had begun.
2. Two sets of parallel lines are standing out from hazy greyness. They are forming the base for division of a plane. The properties of forming figures depend on the angle and the distance in between them. Despite all the changes on those figures, the area of each one will be saved as it is determined by the area of each starting parallelogram.

3. and 4. These two steps are related to a

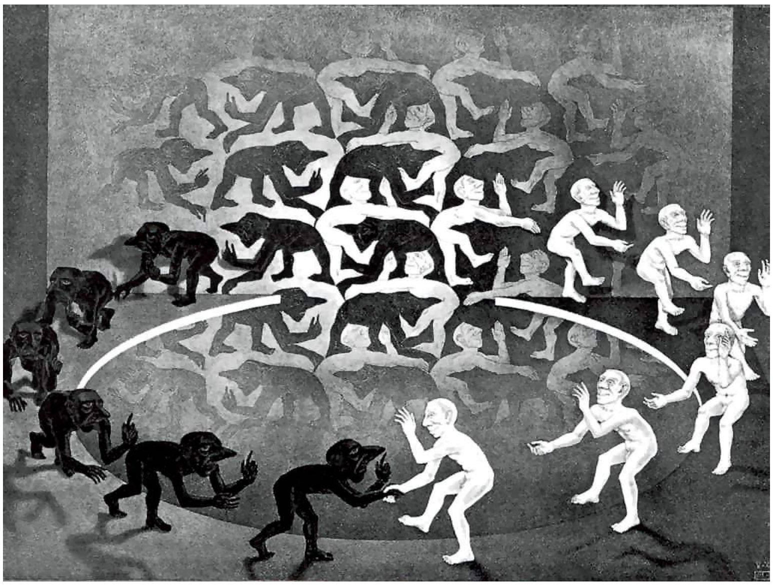
visual partition of neighbouring polygons. The contrast colours are used instead of lines. For the squared division to take place, only two colours are needed – black and white.

5. and 6. The starting point of the fifth step is to choose one of three possible isometrics – translation, rotation or line reflection. Translation is used in the picture ahead of you, as it is the simplest one. Straight border lines on the black and white board are slowly changing in such a way that "where the white colour spreads, the black one decreases."
7. In this step, figures have got their final shape. Though it seems that nothing has left from the starting parallelogram, its properties have stayed completely unchanged and have become parts of a new figure – the area of each motif is equal to the parallelogram and the vertices have not moved at all. The observer might have a blurred impression of a floating thing that has a body and two partly overlapping fins, as if it was a fish or a bird.
8. , 9. and 10. Any doubt about the figure ends here. First, some white details have been added in the black area, and then, in reverse-black details have been added in the white one which all results in combining the two.
11. One can easily notice how black and white figures transform from birds to fish in this step.
12. Finally, it is possible to combine these two sorts of animals in one area.

The steps explained above serve as a proof that mathematics not only deals with numbers but helps create art, as well. The fact that Escher had the same goals as such a magnificent mind as Sir Roger Penrose's is helped him create wonderful pieces of art thus making him one of the best known graphic artist of the 20st century.



M. C. Escher - Cycle



M. C. Escher - Encounter

References

[1] Baloglou, G. (2007) *Isometrica –a geometrical introduction to planar crystallographic groups,* Suny Oswego

[2] Escher, Maurits K. (1998) *Istraživanje beskonačnosti*, IP Esotheria, Beograd.

[3] Escher, M. C.(2008) *The Graphic Work – Introduced and explained by the artist*, Taschen, Koln

[4] Feynman, Richard P. (1999) *The Pleasure of Finding Things Out: The Best Short Works of Richard P. Feynman.* Perseus, Cambridge, MA

[5] Penrouz, R. (2004.) *Carev novi um*, Informatika, Beograd

[6] Petković, M. (2008.) *Zanimljivi matematički problem velikih matematičara*, Društvo matematičara Srbije, Beograd

[7] Simić, S. (2011/12.) *Periodično popločavanje ravni*, Tangenta - časopis za matematiku i računarstvo Društva matematičara Srbije – broj 68/4, 1-8.

[8] http://www.geogebra.org/

[9] http:// www.mathworld.wolfram.com/

[10] http://www.mcescher.com/

[11] http://www.youtube.com/watch?v=T6L6bE_bTMo.