ADVENTURES ON PAPER

Math-Art Activities for Experience-centered Education of Mathematics

Edited by Kristóf Fenyvesi, Ilona Oláhné Téglási and Ibolya Prokajné Szilágyi

Visuality & Mathematics

Experiential Education of Mathematics through Visual Arts, Sciences and Playful Activities

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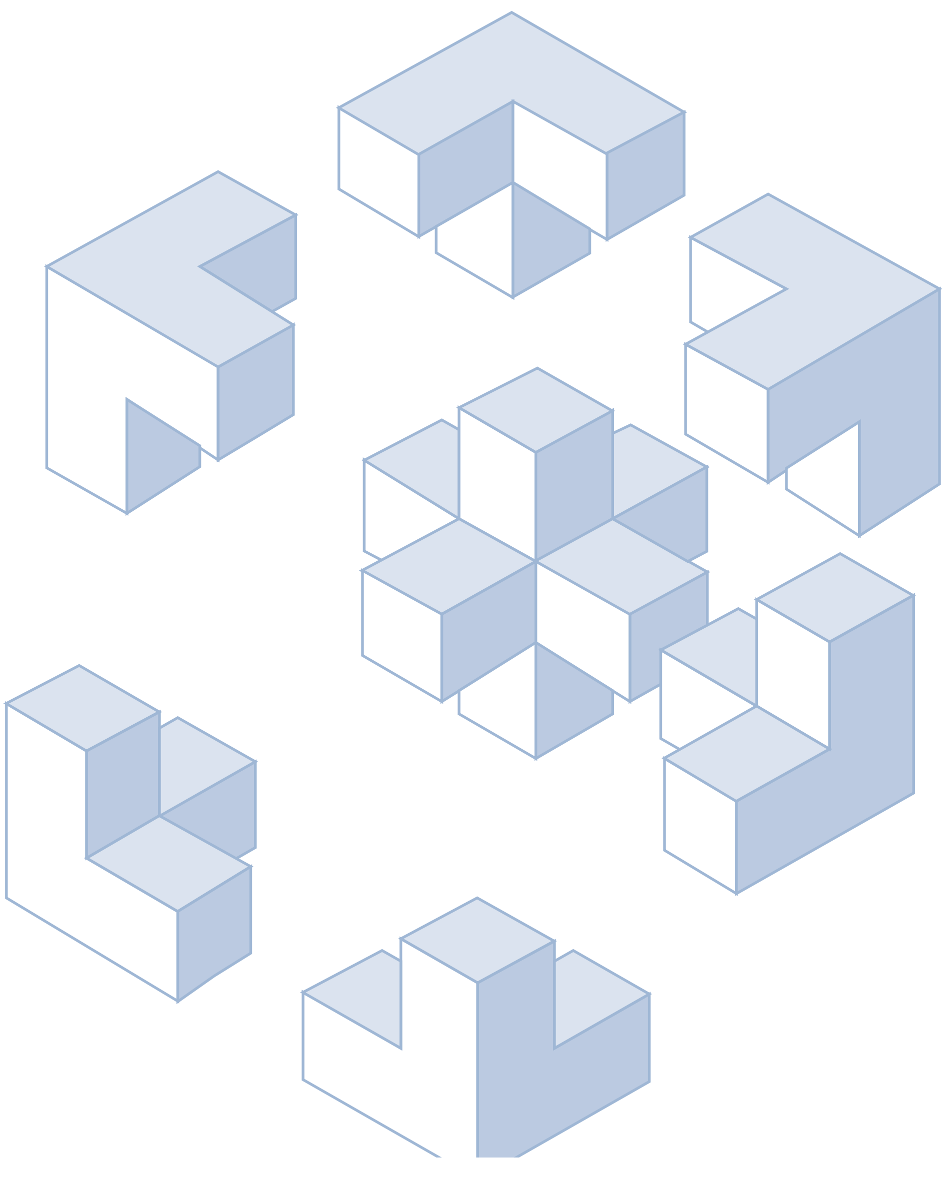

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Edited by Kristóf Fenyvesi, Ilona Oláhné Téglási and Ibolya Prokajné Szilágyi

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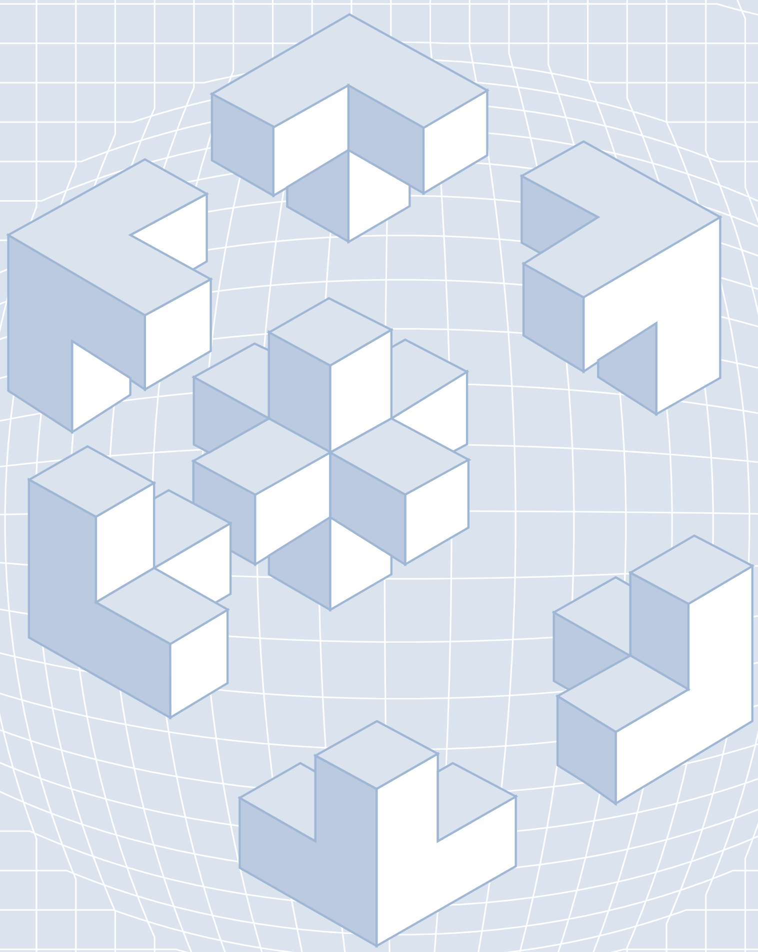
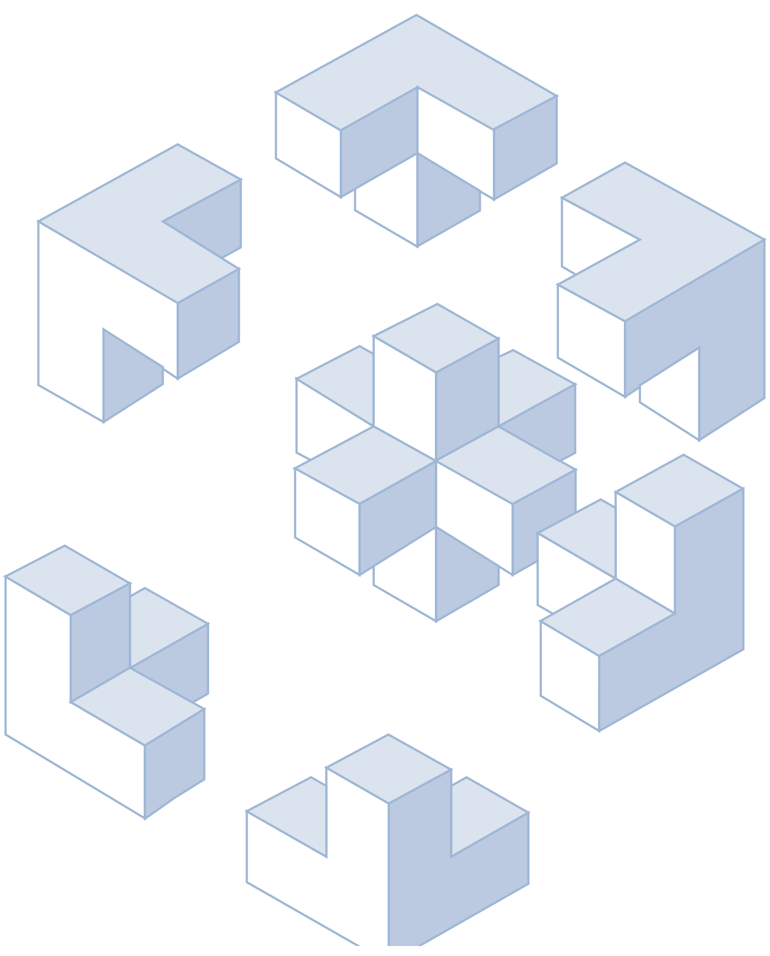




Kristóf Fenyvesi-Ilona Oláhné Téglási-Ibolya Szilágyi:

INTRODUCTION

1 Adventures on Paper – Not Only for the Math Class!

The *Visuality & Mathematics – Experimental Education of Mathematics through Visual Arts, Sciences and Playful Activities* Tempus project was supported by the European Union and launched in 2012 with the cooperation of eight institutions in Austria, Belgium, Finland, Hungary and Serbia. One of the main objectives of this two year international joint effort was to develop educational materials, methods and learning tools that contribute to experience-centered teaching of mathematics. With an interdisciplinary team of mathematicians, artists, educationalists, teachers from both the secondary and tertiary level and university students, our pedagogical aim was to bring about a reawakening of the connections between mathematics and the visual arts. For this purpose we recommended various art-based activities and tasks of a playful and creative nature for use in the math classroom.

This publication is a pedagogical toolkit, which presents hands-on materials and detailed methodological descriptions for the realization of almost forty interactive math-art workshops in the classroom. A number of significant international representatives of visual mathematics, mathematical art and experience-centered mathematics teaching have contributed and we have tried to cover a wide variety of topics. Most of the activities we selected can be completed with paper or cardboard on a cost-effective way and equipment readily-available in most educational institutions such as scissors and colouring pencils, and, at most, a photocopier.

We hope that the teachers who use our toolkit will succeed in encouraging their students to see mathematics as a joyful endeavour rather than the dry and boring subject many believe it to be.[1] Moreover, we hope that this collection is adopted as a handbook not just by math teachers, but also by art teachers wishing to illuminate the links between the two disciplines of art and math.

2 New Trends in Education: Competency and Experience

Society moves at such a pace today, that it is essential our schools equip students with the competencies required to embrace change, to cooperate, to approach challenges successfully at both an individual and collective level and to be prepared for life-long learning. Mathematical competency is one of the key competencies needing to be improved in public education. However, teaching mathematics in elementary and secondary schools purely as an abstract science, utilising only traditional tools and methods has become outdated. Mathematics, like all the other sciences is developing so fast that it is impossible to integrate into the curriculum all the developments taking place in the field. There is also a paradox in that mathematics, although widely used in all industrialized societies, is experienced by most school pupils as a difficult and unpleasant subject.[2] Changes in the approach to mathematics education are therefore overdue and, more importantly, absolutely vital.

Math education in schools should develop systems thinking and be based on applicable knowledge, experience-centered approaches and, above all, the learning process needs to become a more joyful and appealing experience. At the same time, however, learning content cannot be in contradiction with the state of the art mathematics as a science. To implement all of this into everyday teaching practice is not an easy task, yet it is a realizable proposition. This exercise book is an attempt to offer one such alternative contribution, by engaging and familiarizing learners with mathematical contents in an active, hands-on way while revealing mathematical connections in visual arts through creative, playful and engaging activities.

1 Cf. Fenyvesi, Mateus-Berr, Koskimaa, Radovic, Takaci, Zdravkovic (2014) *Serbian Students' Attitudes towards Mathematics and Mathematical Education. Tempus Attitude Survey (TAS) 2013-2014 Report.* Jyväskylä: University of Jyväskylä.

2 Cf. Rogerson, A. (1986), MISP - A New Conception of Mathematics, *International Journal of Mathematical Education in Science and Technology,* 17. 5.

The visual, playful and experience-centered approach to mathematics learning has a rich methodological background and history, beginning with the work of Jerome Bruner[3], Martin Gardner, Zoltán Dienes[4], or Tamás Varga[5]. Recently international interdisciplinary communities such as the *Bridges Organization* (www.bridgesmathart.org), the *International Symmetry Association* (www.symmetry.hu), the *Experience Workshop – International Math-Art Movement* (www.experienceworkshop.hu) and others organizations have done great work in bringing this approach to a larger audience. The importance of research and education programs connecting art and science and their possible role in offering useful new ideas and inspiration for teachers cannot be underestimated.

Providing sufficient motivation for students is maybe one of the greatest challenges in education today as the young are increasingly exposed to a multitude of stimuli. But endearing students to the subject is one of the keys to their learning mathematics well.. This book is devoted to furthering this cause. It was the Romanian mathematician, Grigore Moisil who said: "A mathematician is concerned with mathematics because she sees something beautiful in it, something interesting which she likes, which makes her think and which carries her away. Imagination is a source of information itself."[6] Let your students use their imagination! Encourage students to recognize patterns, to understand systems and to think mathematically, through creative and playful activities!

3 Contents

Our handbook covers fifteen topics in all, arranged into four chapters. The first chapter, *Puzzling Symmetries* opens with Slavik Jablan and Ljiljana Radovic's article and workshop material. The authors are internationally renowned representatives of the interdisciplinary areas of visual mathematics and symmetry studies, as well as knot theory. Their article provides characteristic evidence on the exceptional educational potential of their main research fields, and their discussion concerning symmetry studies can serve as a theoretical introduction to all the topics in our book. A former NASA engineer and presently the art curator of the Bridges Conferences Robert Fathauer's article follows the path of one of the most famous mathematical artists, M. C. Escher, and links the topic of planar tessellations with research on spatial solids. Specialist of playful mathematics education and competency development, teacher at Eger University and one of the editors of our book, Ilona Oláhné Téglási's article utilize tangram- and match-stick games, which are easy to execute in the classroom, for improving several competency-groups. Next Eleonóra Stettner, a mathematician and research coordinator at the Experience Workshop, invites students into the world of graphs and frieze symmetries with her special patterned Moebius strips.

The second chapter *Arabesques and Quasicrsystals* begins with the math-art-education tools developed by Jay Bonner, a renowned expert in Islamic ornamentation and design, and a leading restorer who has worked on the conservation of several historical buildings in the Islamic world. Bonner lets the reader in on the secrets of creating complex Islamic geometric ornamentation and patterns. Jean-Marc Castera from France, a designer and leading art specialist in several architectural projects in the Middle-East expands on Bonner's activity and describes further methods of Islamic pattern design and links it with research on crystallographic studies on quasi-crystals. Ancient Persian art authority and president of the Bridges Organization, Reza Sarhangi presents the application of Islamic patterns on the surface of some special solids.

At the beginning of the third chapter *Tricky Structures, Playful Perspectives*, the artist Tamás F. Farkas shares the geometrical knowledge behind his artistic "impossible figures", which are worth studying from both an aesthetic and scientific viewpoint. F. Farkas' piece enables the reader to show that creating "impossible" figures is not impossible at all and it is not exceptionally difficult, if someone has the requisite mathematical knowledge. Georg Glaeser, mathematics professor and photographer, author of several scientific and popular math books on visual representations of mathematics, together with the artist Lilian Wieser, invites students on an extraordinary adventure: everybody can speculate and test for themselves the mathematical knowledge required for the Nazca people to create gigantic figures

[3] Bruner, J. (1977), *The Process of Education,* Harvard University Press.
[4] Dienes, Z. P. (2002), *Mathematics as an Art form. An essay about the stages of mathematics learning in an artistic evaluation of mathematical activity.* Retrieved from <http://www.zoltandienes.com/wp-content/uploads/2010/05/Mathematics_as_an_art_form.pdf>
[5] Varga, T.-Servais, W. (1971), *Teaching School Mathematics,* Penguin, Harmondsworth.
[6] *Aforizmák, anekdoták matematikusokról, matematikáról* (1999), ed. Bitay, László. Kolozsvár: Radó Ferenc Matematikaművelő Társaság.

in the Nazca desert in southern Peru. Kristóf Fenyvesi, one of the book's editors, coordinator of several math-art organizations and researcher at the University of Jyväskylä, along with the math teacher and pedagogical coordinator of the Experience Workshop Ildikó Szabó get students to build a large model of Buckminster-Fuller's famous geodesic dome. In their article they also introduce the methods of creating anamorphosic images in the classroom. Mathematics professor and noted mathematics populariser, Dirk Huylebrouck, the Belgian coordinator of our *Visuality & Mathematics* project, provides the "magic formula" and requirements for building a three-dimensional fractal tree, as well as a Sierpinski-pyramid from balloons.

In the final chapter, *Paper Sculptures,* mathematics professor and genuine "star" of the international math-art scene and one of the directors of the Bridges Organization, George Hart, familiarizes readers with three-dimensional structures created by paper modules in an ingenious and simple way. Hart's "slide-togethers" are followed by the rich material from the Dutch mathematical sculptor Rinus Roelofs, who offers specific solutions for three-dimensional modelling of figures designed by one of the greatest math-art genius of all-time Leonardo Da Vinci. Turkish mathematician and artist Ferhan Kiziltepe demonstrates how to make highly aesthetic minimalistic art compositions based on very simple geometrical procedures.

Of course, we couldn't possibly have a book espousing the math-art connection without at least one activity involving origami. The mathematical connections in paper folding and origami's great potential in mathematics education are widely known and accepted. The closing article of our collection allows shines a light into all of this, based on the materials and methods of Polish origami artists and internationally recognized math education specialists Wojtek and Krystyna Burczyk.

4 How to use this book?

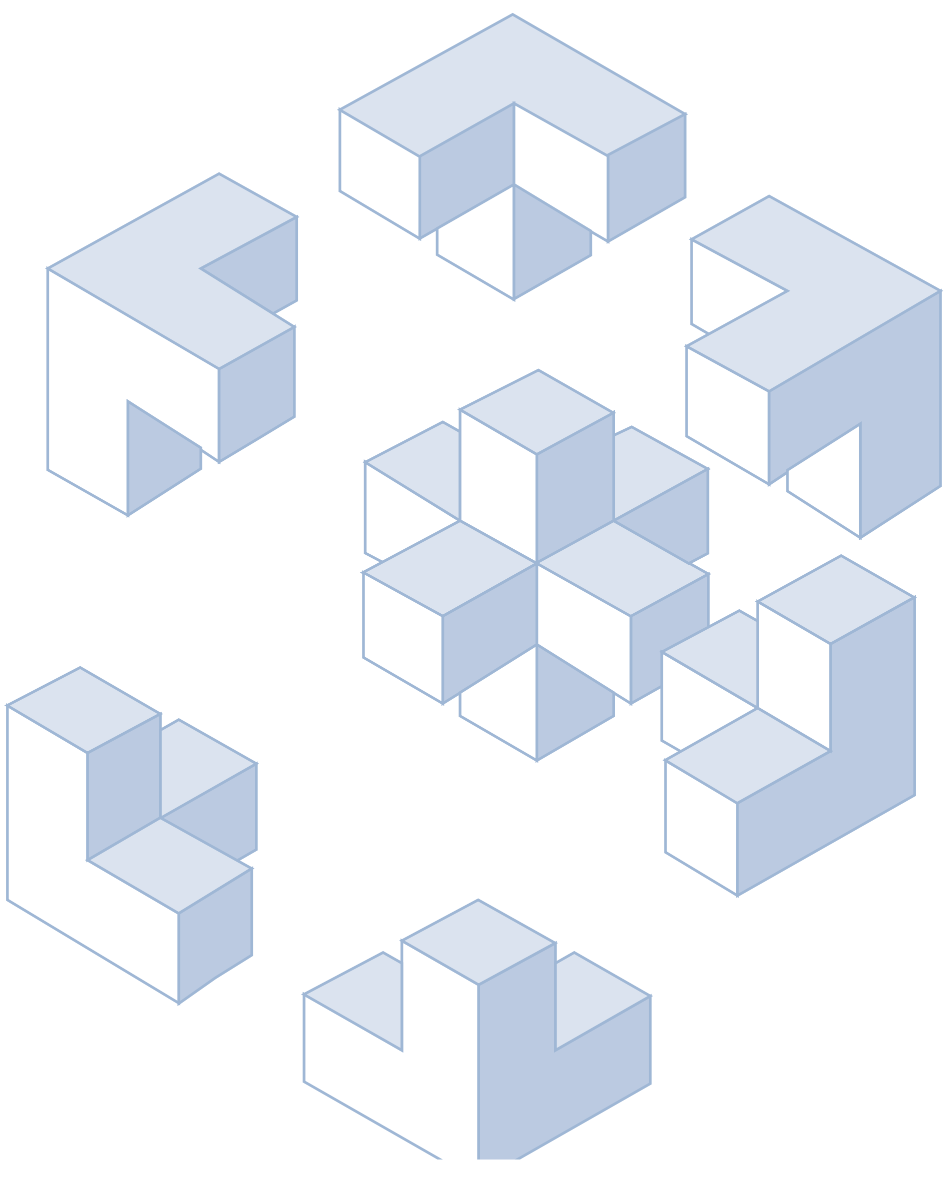
Thanks to the unique features of the book, all of the workshop templates are printed on pages that can be put into a photocopier and copies given to the students participating in the particular workshop. It is necessary for the teacher not only to be immersed in the topic of the selected workshop but also to plan the accompanying activities carefully and to try and test each of the planned workshops and required procedures before holding the selected workshop. During the workshop activities, it is important to plan fewer, rather than more exercises to be covered in a given period of time and to let students enough space and time to discuss. Moreover, it is important to plan and direct workshops according to didactical considerations of *inquiry based mathematics education*[7], and experience-centered education of mathematics[8], to create an atmosphere required for productive learning and to facilitate creativity, to motivate teamwork and collective discussions and to help students possibly lagging behind with patience and encouragement.

Computer applications to complement our handbook's hands-on materials with exercises in digital problem-solving can be downloaded from the following page: http://vismath.ektf.hu/exercisebook

5 Acknowledgements

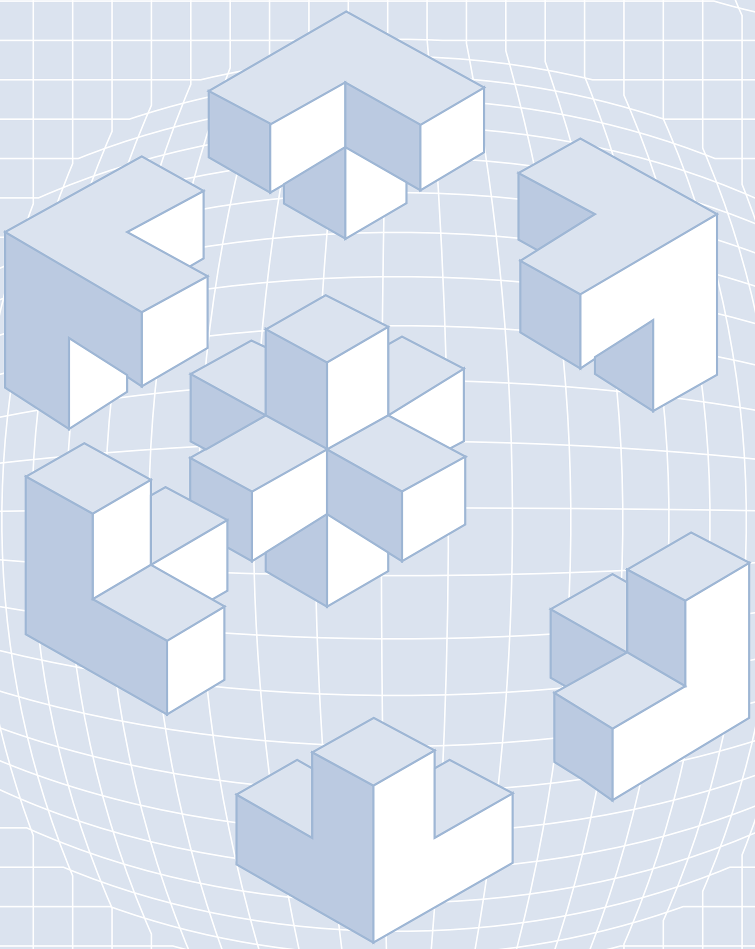
We would like to thank the representatives of the eight partner institutions taking part in the Tempus programme for their constructive work. Many thanks to our collaborators at Belgrade Metropolitan University, ICT College of Vocational Studies, University of Applied Arts Vienna, University of Jyväskylä, University of Novi Sad, Serbian Academy of Sciences and Arts, Sint-Lucas School of Architecture, and colleagues at the consortium leader Eszterházy Károly College, Hungary.

[7] Artigue, M., & Blomhøj, M. (2013), *Conceptualising inquiry based education in mathematics*, ZDM—The International Journal on Mathematics Education, 45(6). 901-909.
[8] Fenyvesi, K. (2012) The Experience Workshop MathArt Movement: Experience-centered Education of Mathematics through Arts, Sciences and Playful Activities, *Proceedings of Bridges 2012 World Conference.* Baltimore: Towson UP, 239-246.



Slavik V. Jablan
Ljiljana Radovic

Patterns, Symmetry, Modularity and Tile Games



Visuality & Mathematics
Experiential Education of Mathematics through Visual Arts, Sciences and Playful Activities

Puzzling Symmetries

SLAVIK V. JABLAN (born in Sarajevo on 10th June 1952) graduated in mathematics from the University of Belgrade (1971), where he also gained his M.A. (1981), and Ph.D. degree (1984). Participated in the postdoc scientific programs in Kishinev (Moldova, 1985), USA and Canada (1990). Fulbright scholar in 2003/4. Published a few monographs (*Symmetry, Ornament and Modularity*, World Scientific, New Jersey, London, Singapore, Hong Kong, 2002;) *LinKnot-Knot theory by computer*, World Scientific, Singapore (2007) and webMathematica book *Linknot* (http://math.ict.edu.rs/), more than 70 papers on the knot theory, theory of symmetry and ornament, antisymmetry, colored symmetry, and ethno-mathematics, and participated at many international conferences (Bridges, ISIS Symmetry Congresses, Gathering for Gardner, knot-theory and mathematical crystallography conferences) and created visual-mathematics course at Belgrade Metropolitan University. The co-editor of electronic journal "VisMath" (http://www.mi.sanu.ac.rs/vismath/). As a painter and math-artist he has more than 15 exhibitions under his belt and an award from the International Competition of Industrial Design and New Technology CEVISAMA-'87 (Valencia, Spain).

LJILJANA M. RADOVIC (born in Nis, Serbia on 28th October 1969) graduated in mathematics at the University of Nis (1993), where she also gained her M.Sc. in Mathematics (2000), and Ph.D. degree (2004) supervised by Professor Slavik Jablan. Works as Associate Professor at the University of Nis, Serbia. Published more than 30 papers on the theory of symmetry and ornament, colored symmetry and antisymmetry, ethno-mathematics, the knot theory. Participated at many international conferences (Bridges, ISAMA, ISIS Congress, Geometry Congresses). Participated in creating visual-mathematics course at Belgrade Metropolitan University with Professor Slavik Jablan. Co-author of the book *The Vasarely Playhouse* (Association for South-Pannon Museums, Hungary, 2011) and the co-editor of the book *Experience-centered Approach and Visuality in the Education of Mathematics and Physics* (Kaposvar University, Hungary, 2012). The co-editor of electronic journal "VisMath" (http://www.mi.sanu.ac.rs/vismath/). Participated in several research projects in geometry, education and visualization.

Patterns and Symmetry

A *pattern* is a discernible regularity in the natural world or in a manmade design, where the elements of pattern repeat in some predictable manner. In the world of patterns we can distinguish and classify patterns according to the types of regularities they use, but also according to the degree of freedom.

The most usual patterns originate from symmetry: the regular use of geometrical rules, based on repetition. It is possible to find symmetry in nature almost everywhere, so we can say that ”nature loves patterns”. The reason for this is the universal principle of economy in nature: a single element (e.g., a tile) is periodically repeated, and as a result produces ornamentation (lat. *Ornamentum* – decoration), one of the oldest and basic decorative elements in visual arts. Artists and artisans of different epochs, cultures and civilizations used the repetition and combination of motifs for the creation and construction of different decorative patterns on bone, textile, ceramics, paintings, and jewelry. Throughout human history, there have existed unbreakable connections between geometry and art, where the visual presentation often served as the basis for geometrical consideration. This applies especially to ornamental art, referred to by Hermann Weyl[1] as ”the oldest aspect of higher mathematics given implicitly”. However, the relatively independent development of geometry and painting resulted in the formation of two different languages, using completely different terms for describing symmetrical forms. In the 20th century, the dynamic progress of the mathematical theory of symmetry meant that the first, more significant impulse for the study of ornamental art came from mathematicians. In the appendix of his monograph about infinite groups, Andreas Speiser[2] proposed using ornaments from the Ancient world (like Egyptian ornaments) as the best possible illustration of symmetry groups. The approach to the classification and analysis of ornaments based on symmetries was enriched by the contributions of different authors (A. Müller, A.O. Shepard, N.V. Belov, D. Washburn, D. Crowe, B. Grünbaum). In their works, descriptive language was replaced by geometric-crystallographic terminology. The approach to ornamental art from the point of view of the theory of symmetry offers possibilities for the more profound study of the complete historical development of ornamental art, regularities and laws on which constructions of ornaments are based, as well as an efficient method for the classification, comparative analysis and reconstruction of ornaments.

The basic, simplest symmetry which generates all the other symmetries is a *mirror reflection* (or just: reflection): a mirror image. We are mirror symmetrical, most living creatures and human creations are mirror symmetrical. Our mirror image is congruent to us, but the orientation is changed: the ”left” figure is transformed into ”right”, where all distances between pairs of points remain unchanged.

A *mapping* (or *transformation*) assigns to every point of the plane its image and *vice versa*. An *isometry* is a transformation which preserves distances between points.

The simplest examples of isometries are the motions of a rigid body: if you move it, all distances between pairs of its points (including the complete shape) will remain unchanged. We say that a figure is *invariant* under the action of some transformation if it remains unchanged under this action. For example, if you rotate a square around its center by 90 , or if you make its mirror image in a vertical or horizontal symmetry axis, or its diagonal, the square maps to itself. Such a transformation (in our example the rotation or mirror reflection) is called the *symmetry* of this figure. The concept of symmetry exceeds the domain of isometries, and it is related to the invariance with regard to arbitrary transformations (e.g., similarity symmetries, *etc.*).

Each figure possesses at least one (trivial) symmetry, the so-called *identical transformation*, leaving unchanged (invariant) every point of the figure. A figure with a single symmetry – the identical transformation, is called asymmetrical; otherwise, it is called symmetrical. For example, the capital letters A,B,C,D,E,K,M,T,U,V,W,Y are mirror symmetrical with regard to the vertical axes; H,I,O,X are double-symmetrical (in vertical and horizontal mirror) and centro-symmetrical, and F,G,J,P,Q,R are asymmetrical. Letters bd or pq are mirror-symmetrical pairs, and bq or pd are centro-symmetrical pairs. In a similar way, in the sequence every letter can be moved by one step, or by two steps, and this is a translation. The minimal translation is one-step translation.

Mirror reflections generate all other isometries. Every plane isometry can be represented as the composition of at most three mirror reflections. Because each reflection has its own axis, the problem of the classification of plane isometries can be reduced to the question about possible mutual positions of three lines in the plane (reflection axes) and resulting isometries (Fig. 1). If the two reflections are parallel, their composition is a *translation* (parallel motion).

[1] H. Weyl, *Symmetry*, Princeton, 1952

[2] A. Speiser, *Theorie der Gruppen von endlicher Ordnung*, Berlin, 1927

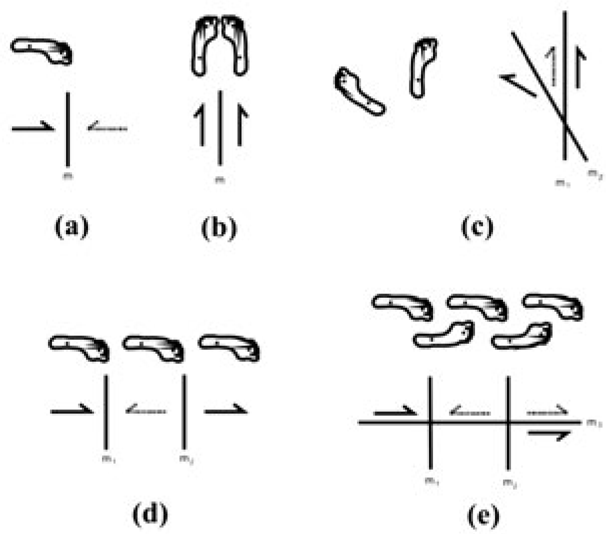


Figure 1: Plane isometries: (a) identity transformation; (b) mirror reflection;
(c) rotation; (d) translation; (e) glide reflection

If the two reflection axes intersect, the result of the composition of these two reflections is a *rotation* around the intersection point. A composition of the three reflections with axes not intersecting in the same point is a *glide reflection*.

Mirror symmetry (i.e., mirror reflection) changes the orientation of every figure: a "left" oriented figure transforms to the "right" oriented and *vice versa*. If we apply a mirror reflection two times, or an even number of times to the same figure, its orientation remains unchanged; if we apply it an odd number of times, the orientation of the object changes, so we can distinguish among plane isometries:

1. even isometries (the sense-preserving transformations): identical transformation, rotation, and translation (the parallel shift) – compositions of two reflections, and
2. odd isometries (the sense-reversing transformations): reflection and glide reflection ("footprints in the snow") the composition of three reflections.

All symmetries of an object make a mathematical structure called *group* (in our case, a symmetry group). All elements of a group satisfy four properties. The first property is the *closeness*: the result of a composition (successive action) of two symmetries is symmetry. The second is the *associativity*: there is no difference if the composition (successive action) of two transformations is followed by a third, or the action of the first is followed by the composition (successive action) of the other two transformations. The third property is *the existence of a neutral element* – a trivial, identity transformation, preserving all points of a figure unchanged. The fourth property is *the existence of the inverse symmetry*: if some figure F is transformed by some symmetry into a figure F_1, then the inverse transformation maps F_1 back to F.

We are interested in the symmetric figures and plane patterns, remaining unchanged by the action of nontrivial transformations – symmetries. We distinguish three kinds of symmetric figures in a plane:

1. the figures which have preserved a single invariant (unchanged) point of the figure (the center) under the action of symmetries – the *rosettes* (circular patterns),
2. the figures without the invariant points, with an invariant direction, the translation axis – the *friezes* (linear patterns),
3. the patterns without invariant points, and with two invariant directions – the *plane ornaments.*

To each ornament we will assign the corresponding symmetry group and divide the rosettes, friezes, and ornaments into a (finite) number of classes, using their symmetry groups as the classification criterion. There are two infinite classes of the rosettes: cyclic and dihedral. Every frieze belongs to one of the seven types, and every ornament belongs to one of the 17 types of plane ornaments.

For the recognition and construction of friezes and ornaments we can use flowcharts showing in the schematic way their symmetry structure. As homework you can try to recognize the symmetry of different friezes and ornaments coming from ornamental art, as well as construct different symmetry patterns derived from different initial asymmetrical basic elements.

Figure 2: Flowcharts for the recognition of friezes

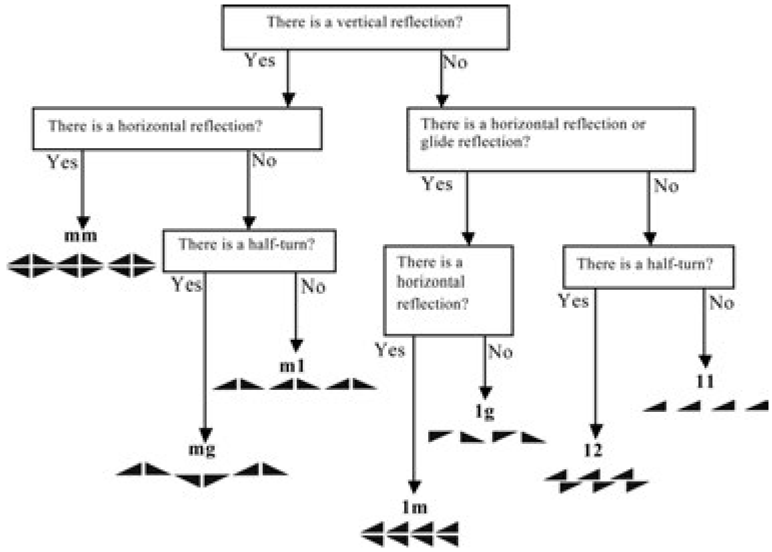
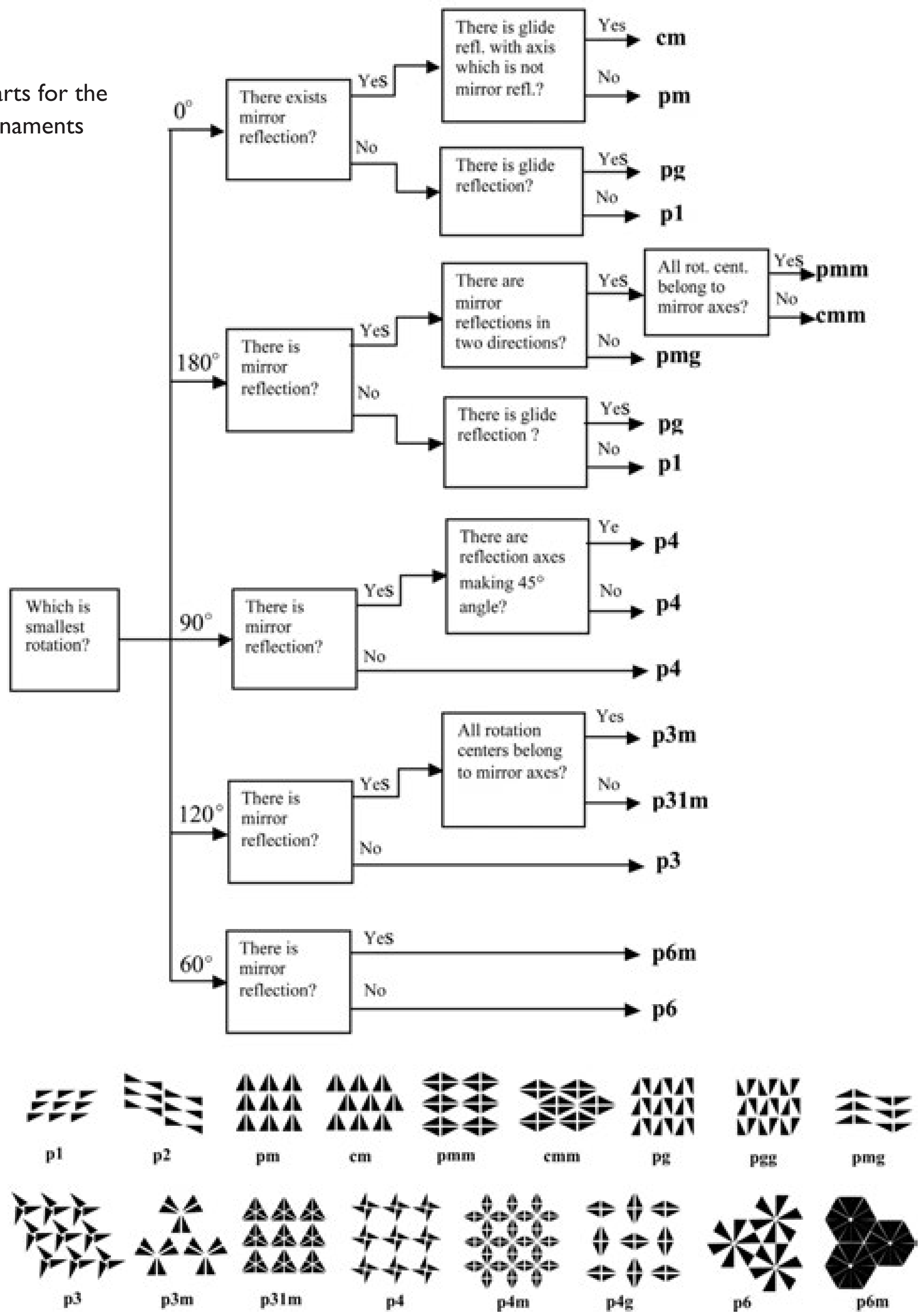


Figure 3: Flowcharts for the recognition of ornaments



Game 1: Symmetry recognition

Try to recognize the symmetry group of the friezes and plane ornaments shown in the Fig. 4 following flow charts instructions.

Figure 4: Examples of friezes and ornaments (Medieval Serbian frescoes).

MODULARITY

As we already mentioned, symmetry is merely the simplest way to create patterns, which, thanks to periodicity, provides the lowest degree of freedom and complexity. The next possibility for economy in nature is the *recombination*. One of its manifestations is the principle of *modularity*: the possibility to create diverse and variable structures, originating from some (finite and restricted) set of basic elements, by their reordering, where from a minimal number of initial elements (modules) we construct the maximum number of possibilites. In this construction it is best to make equal use of the modules. Scientists have always searched for the basic building blocks of nature in physics, chemistry, biology, and the other sciences. In Plato's philosophical treatise "Timaeus", (written in the style of a Socratic dialogue), the four basic elements in nature: earth, fire, air, and water, are identified with the regular polyhedra: cube, octahedron, icosahedron, tetrahedron, and the fifth regular polyhedron, dodecahedron, represented the Universe. In physics, beginning from subatomic particles, atoms, or elementary energy entities, quarks – the units of matter or energy –, scientists try to explain nature by using modularity. A similar tendency occurs in art, architecture and design (especially ornamental).

Further, we will illustrate some examples of modularity, designed as types of modular games, and their potential use in mathematical education. Also, we will explain their origin and show that the principle of modularity has been used from the since the beginnings of human history as the basis for the construction of diverse structures.

The oldest example of modularity based on geometric figures and structures comes from the Paleolithic excavation site, Mezin (Ukraine, 23 000 B.C.). "Op-tiles" are a set of modular elements which consists of two rectilinear antisymmetrical tiles (a positive and negative). It can be extended to the set consisting of five elements, including two curvilinear tiles and one antisymmetrical tile (an "Op-tile" based on the same principle as the Truchet tile, antisymmetric with respect to a diagonal). From these tiles it is possible to construct an infinite collection of plain black-white used in the twentieth century artform known as "Op-art" (optical art). For such a modular element – a square with a set of parallel diagonal black and white stripes and its negative, we also use the name "Versitile", proposed by the architect Ben Nicholson who discovered the same family of modular tiles by analyzing Greek and Roman meander friezes and mazes. Key patterns constructed from Op-tiles produce powerful visual effects of flickering and dazzle, thanks to the ambiguity which occurs due to the congruence between "figure" (the black part) and "ground" (white part) of the key-pattern when our eye oscillates between two equally probable interpretations. This kind of visual effect is abundantly used in "Op-art", which makes use of the visual effects of different geometric forms and colors from the theory of visual perception and visual illusions.

Let’s see how the creative process for the design of this ornament may have developed. Imagine a modern engineer who begins a construction project. At first he makes a rough sketch, and then he begins to work more seriously on solving the problem. The next series of ornaments from Mezin is more advanced. In Figure 5a we see the masterpiece of Paleolithic art – the Birds of Mezin decorated with meander ornamentation. The man of prehistory has applied the symmetry constructions that he learned, and he has preserved them for posterity. On the mammoth bone, modelled in the form of a bird, he engraved the meander pattern which represents the oldest example of a rectilinear spiral in the form of a meander ornamentation.

(a) (b) (c)

Figure 5: (a) Bird of Mezin; (b) Mezin bracelet; (c) developed bracelet

For the drawing of Op-tiles and similar modular elements, today you can use graphic computer programs. Some of the computer drawing programs are free and can be downloaded from the Internet, while for others you will need a license. From the address http://www.inkscape.org/ you can download the free program ***Inkscape***, which can be substituted for ***Corel Draw***. However, if you have a licensed version of ***Corel Draw*** or ***Adobe Illustrator***, you can draw all the illustrations in these programs. After printing the tiles obtained on the transparencies, you can produce different 3D structures, similar to those created by the artist Victor Vasarely, or make the Op-tile game.

Game 2: Op-tile game

From Op-tiles make your own design. Certainly, you can put the pieces together edge-on-edge (Fig. 6a), but more interesting effects you can obtain by shifting (Fig. 6b). The final design glue on the paper. You can also create “Op-tile letters” (Fig. 6c)

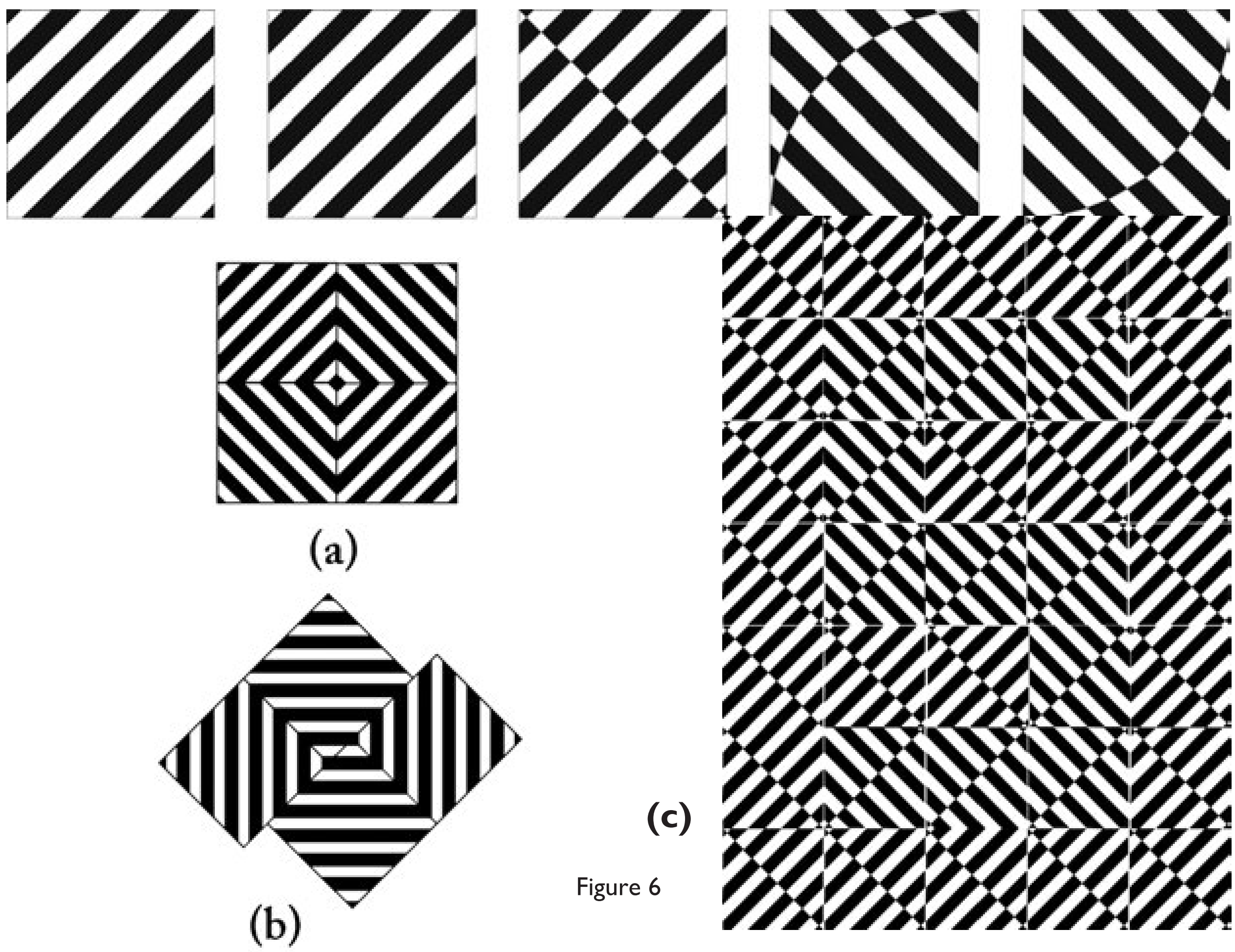


Figure 6

Game 3: Transparent Op-cubes

1. Fold the edges denoted by the broken lines (Fig. 7) and glue together the other two edges by using pieces of adhesive transparent tape.
2. From the obtained parts make the Op-cube (Fig. 8).

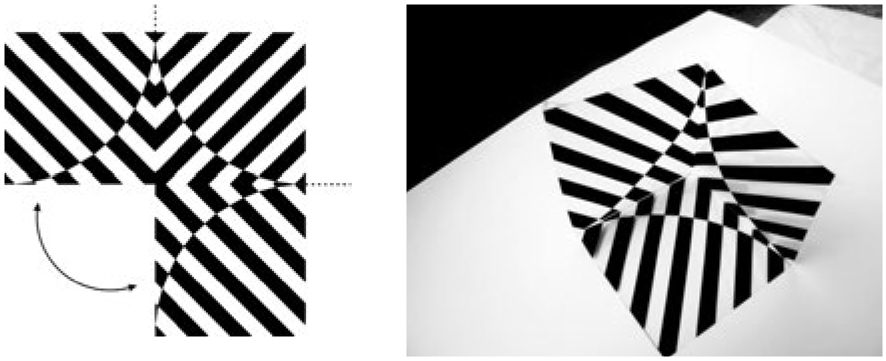


Figure 7

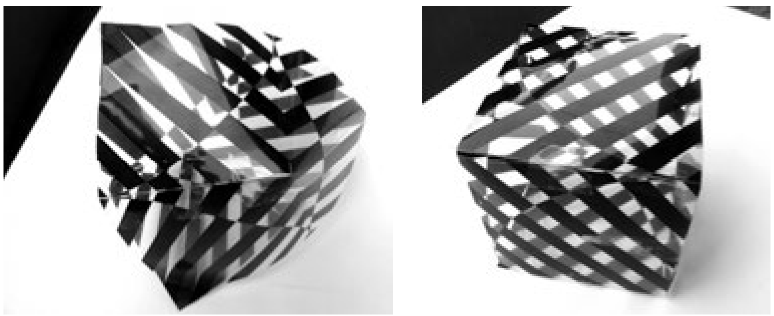


Figure 8

Game 4: Op-tile Hypercube

1. Fold the edges denoted by the broken lines and glue together two opposite edges (Fig. 9a).
2. The resulting 6 pieces should be joined by gluing together the upper edges denoted by the bold lines (Fig. 9b).
3. Make the hypercube from the structure 2) by fixing together the corresponding inside edges by using pieces of adhesive transparent tape (it is not necessary to completely seal the edges, just fix the central parts (Fig. 10).

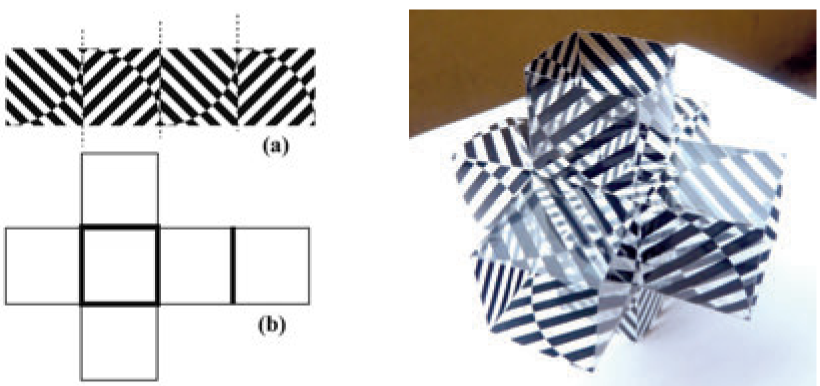


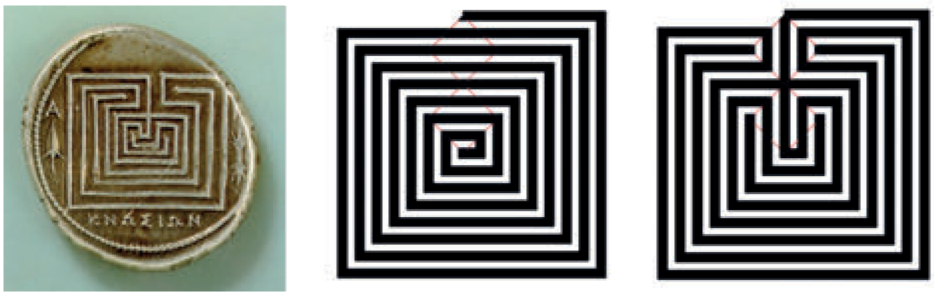
Figure 9

Figure 10

Materials for producing transparent Op-cubes, Op-tile hypercube and Op-tile game, ready for use, can be downloaded from the address: http://www.mi.sanu.ac.rs/vismath/TileGames.zip

LABYRINTHS

The word ”labyrinth“ is derived from the Latin word *labris*, meaning a two-sided axe, the motif related to the Minos palace in Knossos. The walls of the palace were decorated with this ornamentation, while the interior of the palace featured bronze double axes. This is the origin of the name ”labyrinth” and the famous legend about Theseus, Ariadne, and the Minotaur. The Cretan labyrinth is shown on the silver coin from Knossos (400 B.C.) (Fig. 11a). How does one construct a unicursal maze? Figure 11 shows the most elegant way: draw a black meander (Fig. 11b), remove several rectangles or squares, rotate each of them around its center by the 90° angle, and bring it back to obtain a labyrinth (Fig. 11c). Even very complex mazes can be constructed in this way (Fig. 12).



(a) (b) (c)

Figure 11: Top: “Do you like Paleolithic Op-Art?” Slavik Jablan’s exhibit at Bridges Pécs 2010 Conference (Hungary), curated by Kristóf Fenyvesi. Below: (a) Silver coin from Knossos with image of Cretan labyrinth; (b) meander which can be composed from Op-tiles; (c) its transformation to maze

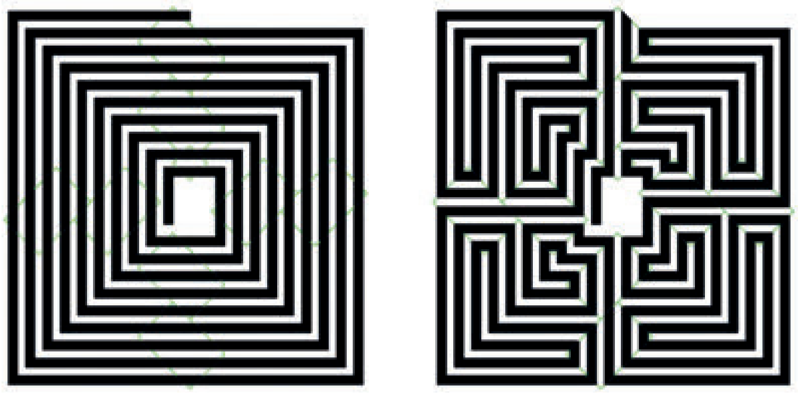


Figure 12: Construction of a more complicated maze

Game 5: Labyrinths exhibition

In larger spaces, you can produce labyrinths by making them from square tiles and plastic adhesive tape (Fig. 13).

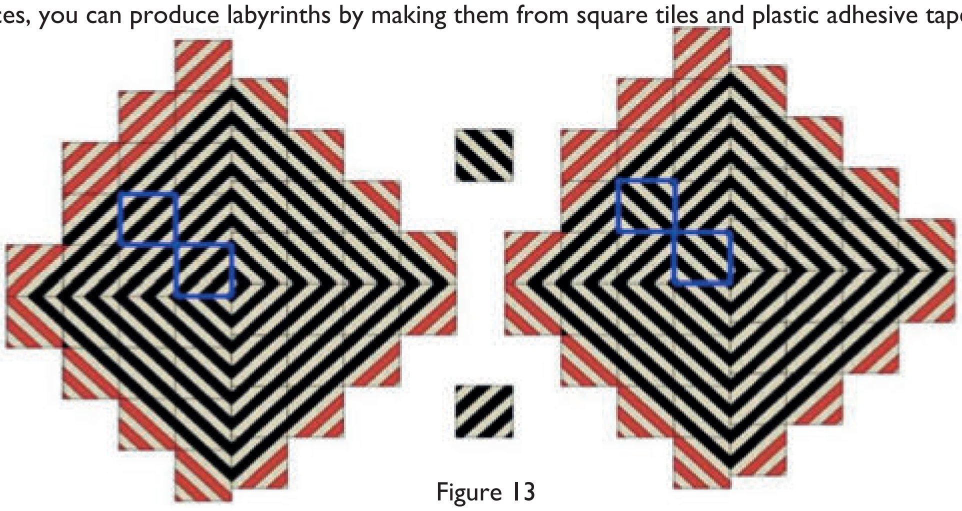


Figure 13

Mirror Curves, Celtic Knots and "Knot-tiles"

Mirror-curves originated from matting, plaiting and basketry. They appear in arts of different cultures (as Celtic knots, Tamil threshold designs, Sona sand drawings etc.), as well as in the works of Leonardo Da Vinci and Albrecht Dürer. Paulus Gerdes recognized their deep connection with the mathematical algorithmic-based structures: knot mosaics, Lunda matrices, self-avoiding curves and cell-automata (find Paulus Gerdes' books at: http://www.lulu.com/spotlight/pgerdes).

Mirror curves can be used for the construction of Celtic knots. Celtic knots are one of the highlights of knot-art. Some researchers believe that the root of Celtic knot art is in knot designs in the 10th-11th century eastern mosaics patterns, especially Persian tiling. The very beginning of knotwork art probably originated in mirror curves constructed from plates, which have been also recognized as the basis of all Celtic knotwork by the archaeologist J. Romilly Allen whose twenty years' work is summarized in the book "Celtic Art in Pagan and Christian Times".

Mirror curves are trajectories of (imaginary) rays of light emitted from edge mid-points in the regular square grid RG[*a*,*b*], with sides *a* and *b*, with one-sided mirrors on their external sides and two-sided mirrors placed between cells, coinciding with internal edges (cell borders) or perpendicular to them at their mid-points. After a series of reflections, a ray of light creates a closed path – a component of the mirror curve. If the complete RG[a,b] is covered by a single mirror curve we have a perfect (monolinear) Sona drawing. Otherwise, after closing the path of one component we start with a new ray of light emitted from another starting point and continue in this way until the whole RG[*a*,*b*] is uniformly covered by mirror curves. How do we draw these curves? Usually we begin with a regular square grid RG[*a*,*b*], but the same can be done with regular triangular or hexagonal grids. In the middle of a square we draw a black dot which will be surrounded by the mirror curve and multiply this square in order to obtain the regular square grid RG[*a*,*b*] (Fig. 14).

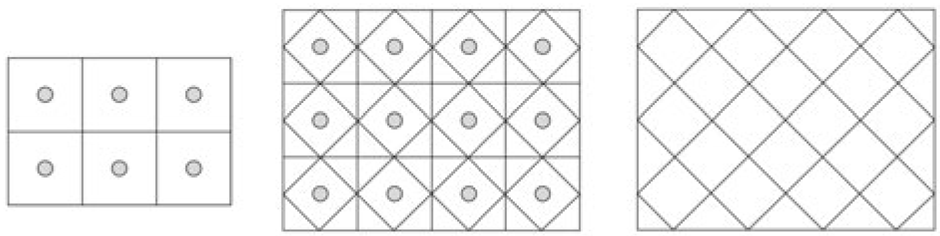


Figure 14: Rectangular grid with monolinear mirror curve

We begin with RG[*a*,*b*] with no internal mirrors. The ray of light reflects and is emitted from the mid-point of an edge, reflects in the mid-points of the external edges of bordering squares, and travels through the grid. In order to simplify this, you can divide RG[*a*,*b*] into small squares, without deleting the internal edges, and remove them after finishing your work. If the numbers *a* and *b* are relatively prime, i.e. if GCD(*a*,*b*)=1 (as in our example where *a*=4 and *b*=3, so *c*=GCD(*a*,*b*)=1), we obtain a single curve surrounding all dots and uniformly covering RG[*a*,*b*]. Otherwise, a multi-component mirror curve is obtained, where its number of components is *c* = GCD(*a*,*b*) . Instead of completely covering RG[*a*,*b*] with dots you can use a system (a scheme) consisting only of dots, but in this case you need to consider the correct direction of the light-ray, especially when turning it in the border ends (Fig. 15).

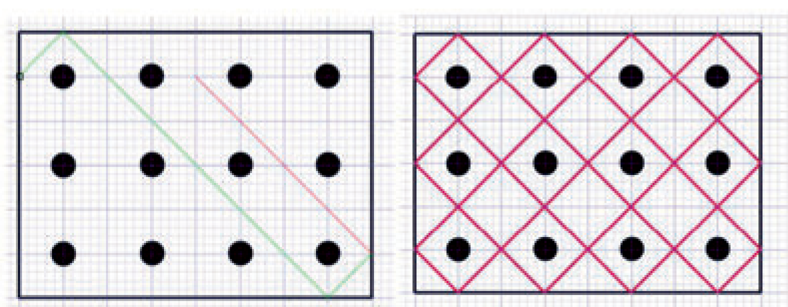


Figure 15

Different situations can occur depending on the dimensions of RG[*a*,*b*], i.e., GCD(*a*,*b*). Try to prove the previous statement that the number of components (it is useful to represent them with different colors, so they can be easily distinguished) is equal to *c*= GCD(*a*,*b*). For example, for *a*=2 and *b*=2 the number of components is *c*= GCD(2,2)=2, for a=4 and b=2 the number of components is c= GCD(4,2)=2, for a=5 and b=2 the number of components is c= GCD(5,2)=1, for a=6 and b=2 the number of components is c= GCD(6,2)=2, for a=6 and b=3 it is c= GCD(6,3)=3 (Fig. 16), *etc.*

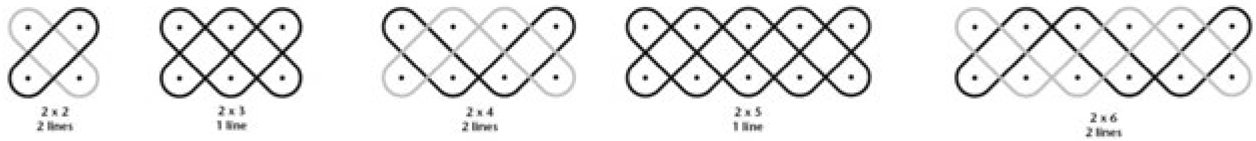


Figure 16

In order to draw curves with smoothed, rounded corners, for reflecting a ray of light in the external edges and in the corners of the grid, you can use the following rules shown in Figure 17 (this can be done in *Adobe Illustrator* by the command Filter>RoundCorners, or by command Effect>Stylize>RoundCorners).

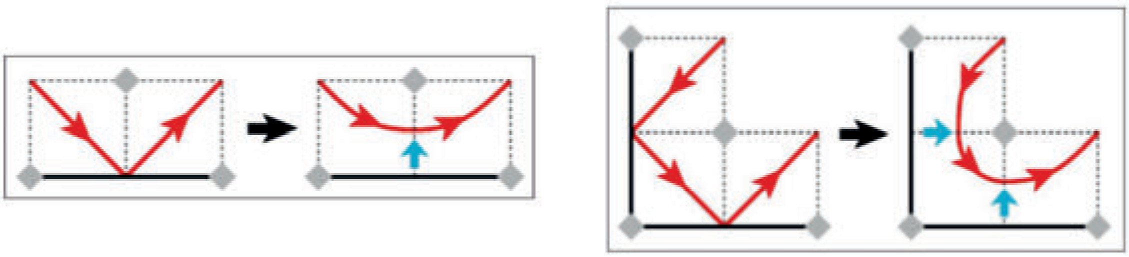


Figure 17

Let's see what will happen if the numbers a and b are not relatively prime. As before, we draw a rectangular grid, but now of dimensions 4 4, and we place the rays of light (components) in it. In order to surround all the dots and uniformly cover RG[4,4] by the mirror curve, we need four components (Fig. 18).

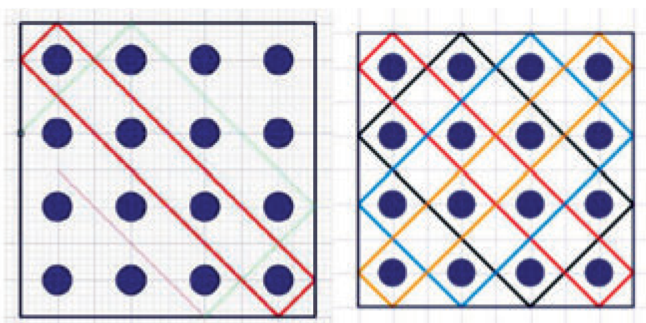


Figure 18

Every mirror curve can be converted into a knotwork design by introducing the relation "over-under". The next question is how to place internal two-sided mirrors within the existing scheme, in order to join our components into a single mirror curve, i.e. to get a single ray of light that traces it. When inserting an internal mirror, we have two possible choices (Fig. 19):

1, place it between square cells, incident to the internal edge;
2, place it perpendicular to the edge at its mid-point.

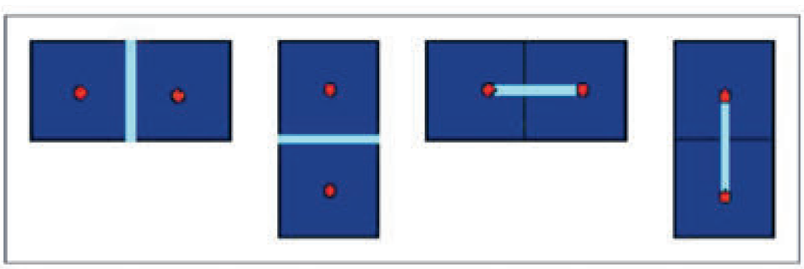


Figure 19

A mirror placed at the crossing point of two components (denoted by different colors) will join them into one. If placed at a self-crossing of an oriented component, a mirror between arrows breaks the component in two; otherwise, after inserting the new mirror, the number of curves remains unchanged (Fig. 20).

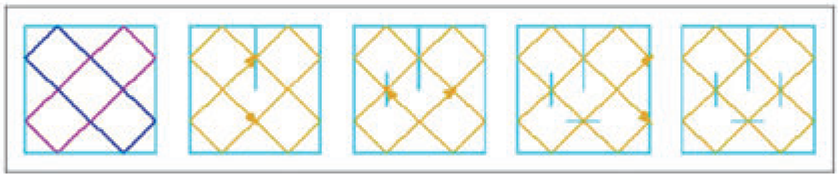


Figure 20

We repeat this algorithm until a single curve is obtained. After inserting internal mirrors, we again make a smoothing of the mirror curve obtained (Fig. 21 left). The next image shows the smoothing of zig-zag lines (Fig. 21 right).

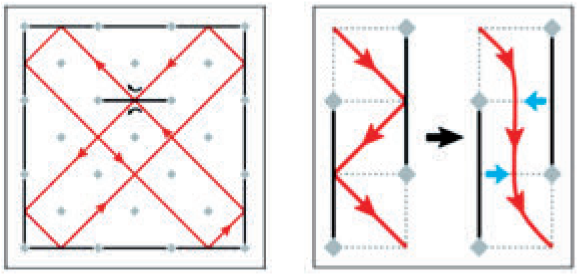


Figure 21

Notice that symmetrical placements of mirrors result in symmetrical mirror curves. Let's return to the example of the 4x4 grid covered by four components. After inserting an internal mirror at the crossing point of two different curves (represented by different colors), the number of components reduces to three, since the two curves are joined into one (after this, you need to make a new coloring of the components, representing the joined curves by a single color). Continuing in this way, by inserting new internal mirrors at the crossing point of two components, at each step the number of curves decreases by one, so that after three steps we obtain a single component – monolinear mirror curve uniformly covering RG[4,4]. Figure 22 shows one possible way to introduce three internal mirrors in order to get a single curve. Certainly, the solution of this problem is not unique; you can choose different arrangements of three internal mirrors each resulting in a monolinear mirror curve.

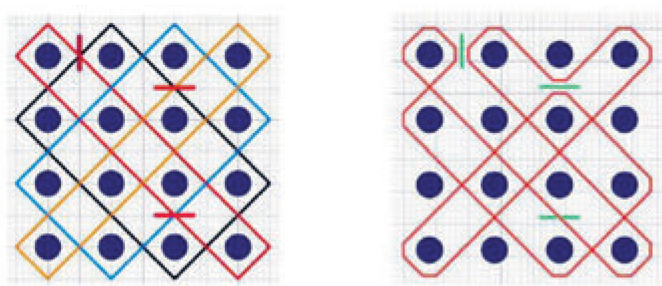


Figure 22

In Figure 22 the mirrors are not symmetric, so the resulting mirror curve is asymmetric as well. Figure 23 shows a symmetric arrangement of four internal mirrors according to a rotation of order 4, so the resulting drawing has the cyclic rotational symmetry of the group C_4. However, we have not completely followed the rule about pairing components of different colors, and so we obtained a 4-component (imperfect) mirror curve. Try to make another symmetrical arrangement of internal mirrors in the same 4x4 grid resulting in a perfect (monolinear) mirror curve.

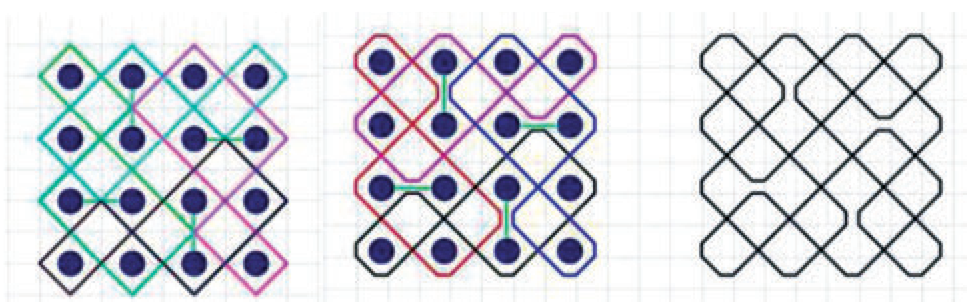


Figure 23: Symmetrical arrangement of internal two side mirrors

Figure 24 illustrates the same exercise within a 3x3 grid. At the beginning, we have 3 components, so it is necessary to insert at least 2 internal mirrors in order to obtain a single curve. By using two internal mirrors, placed asymmetrically, we obtain the drawing shown on the left. The right part of Figure 24 shows the same curve turned into the diagram (projection) of an alternating knot, by introducing at every crossing the relation "over-under" in an alternating manner. You will find many similar examples in books describing Celtic knotwork or in computer programs for drawing Celtic knots.

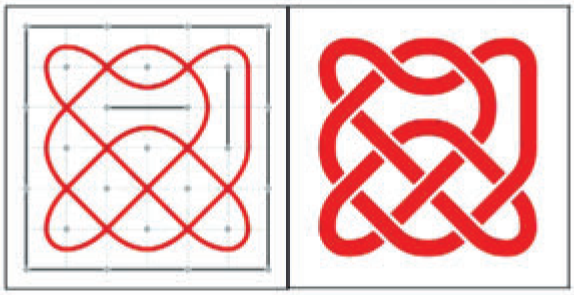


Figure 24: Mirror curve and corresponding knot

In order to obtain interlacing knot ornaments (mirror curves similar to Celtic designs) you can use a few basic tiles (5 modules) as shown in Figure 25. These tiles ("KnotTiles") represent all possible "states" of a single small square with regard to the portion of a mirror curve that it contains along with the relation "over-under". By (re)combining them, you can obtain all possible mirror curves (or knot mosaics). For the knot game made from "KnotTiles", please see http://www.mi.sanu.ac.rs/vismath/op/tiles/kt/kt.htm

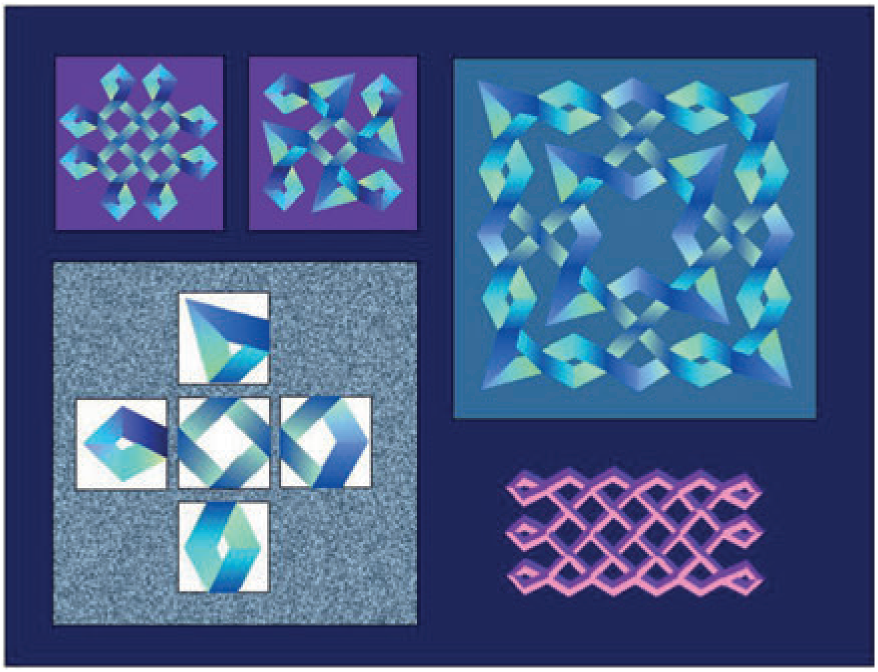


Figure 25

Consider the possibility to insert internal mirrors into a regular square grid lacking internal mirrors and covered by a single curve with c= GCD(a,b)=1, i.e., when the sides of the grid are relatively prime numbers. We can try to plan the choice of positions for mirrors in advance, in order to obtain more interesting and visually pleasing single curves. In many cases this can be achieved by using symmetry (translational = repetitional, rotational, or any other). After some experimentation, you can obtain single curves similar to those created by the Tchokwe people (Fig. 26) from 6x5 square grids (initially containing one component) and 5x5 grids (initially containing 5 components).

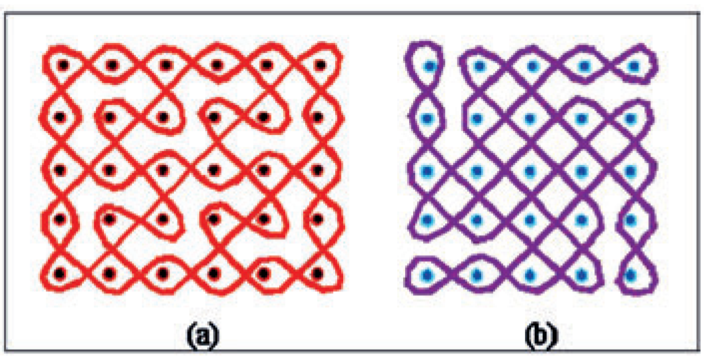


Figure 26

Figure 27 shows the 5x4 grid initially containing a single component mirror curve, but still we can add some internal two sides mirrors. After a few steps, by introducing six internal mirrors in the appropriate (symmetrical) positions, the Celtic people once again obtained a monolinear mirror curve.

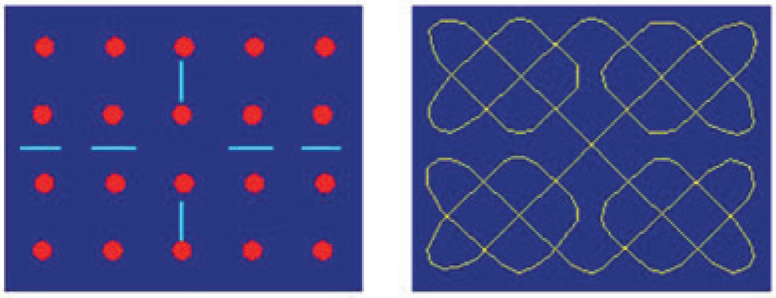


Figure 27

Figure 28 shows examples of Celtic knots from the front cover of "Celtic Art – the Methods of Construction" (Dower, New York, 1973) by G. Bain. All Celtic knots can be (re)constructed by taking a grid (part of a plane tessellation; in the simplest case a part of a square grid), placing internal mirrors in it, and introducing the relation "over-under" (interlacing) at every crossing of the mirror curve.

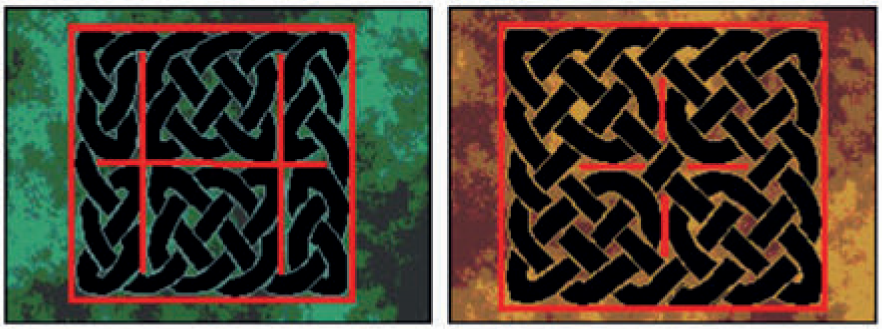


Figure 28

The next knot (Fig. 29) is created in *Adobe Illustrator*. Beginning with the 4x3 grid, we introduce eight internal mirrors and create the mirror curve, drawn by a 20pt line with the use of neon effects. Isn't that simple?

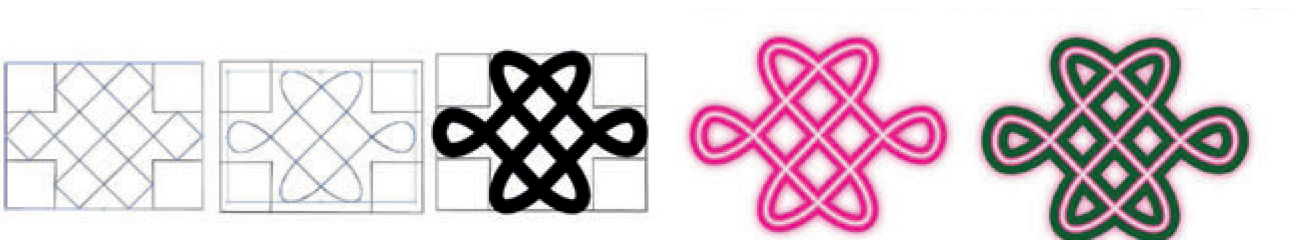


Figure 29

In Figure 30, different mirror schemes are shown. They can be used to create Celtic neon knots in the same manner as we constructed knots in Fig. 29.

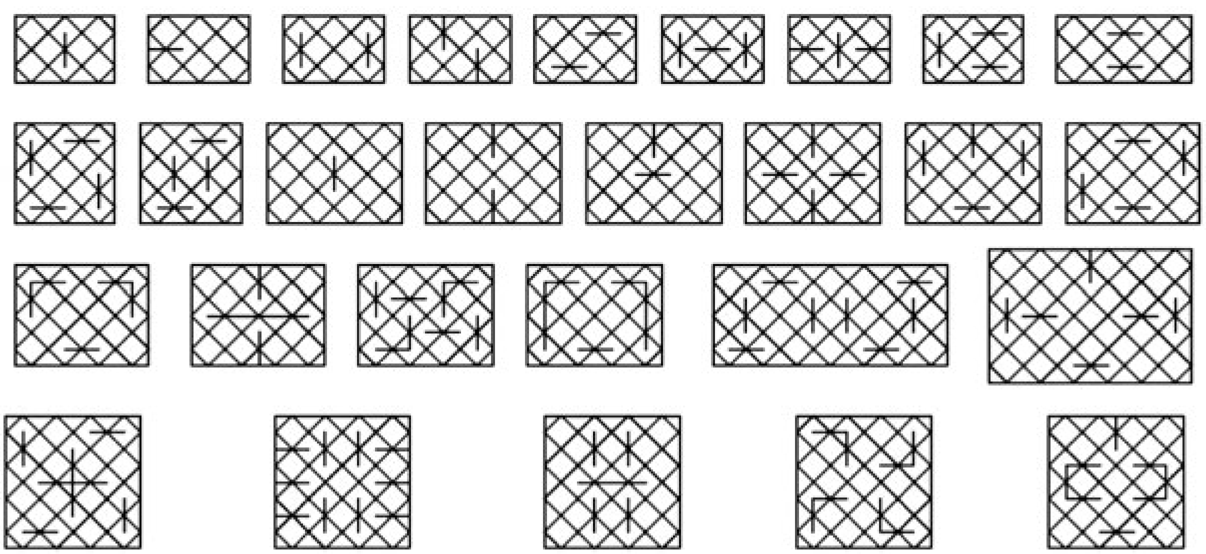


Figure 30

It is also possible to combine several rectangular schemes and extend our mirror-curve designs to obtain friezes or more complicated ornamental knots. By combining, joining, or overlapping parts of basic rectangular grids, we obtain composite mirror-curve designs (Sona drawings of animals made by the Tchokwe people, or monolinear Tamil threshold designs, called "pavitram" or "Brahma mudi") (Fig. 31).

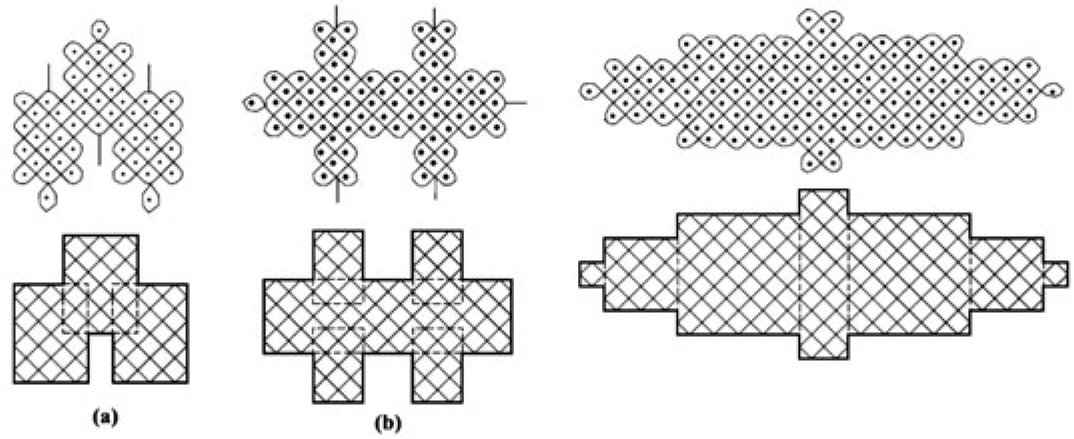


Figure 31

The same approach can be used in order to obtain composite Celtic knots (e.g., in the form of a cross) (Fig. 32)

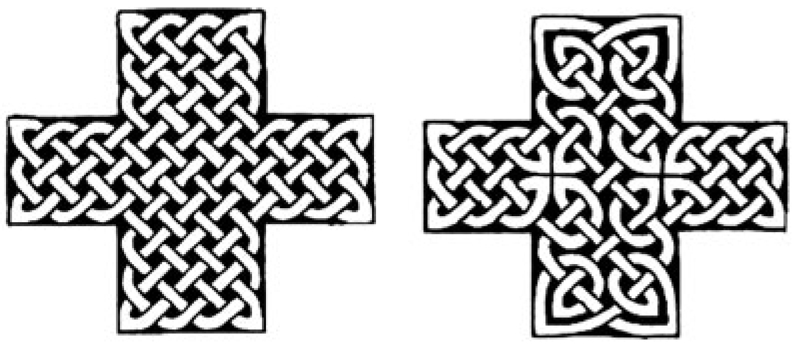


Figure 32

Game 6: Knot tile

Every knot or link you can make from these five Knot-tiles (Fig 33, left). A set of knot-tiles is called perfect if it contains tiles of every of five kinds. Goal of the game: make a knot or link from all 9 knot-tiles.

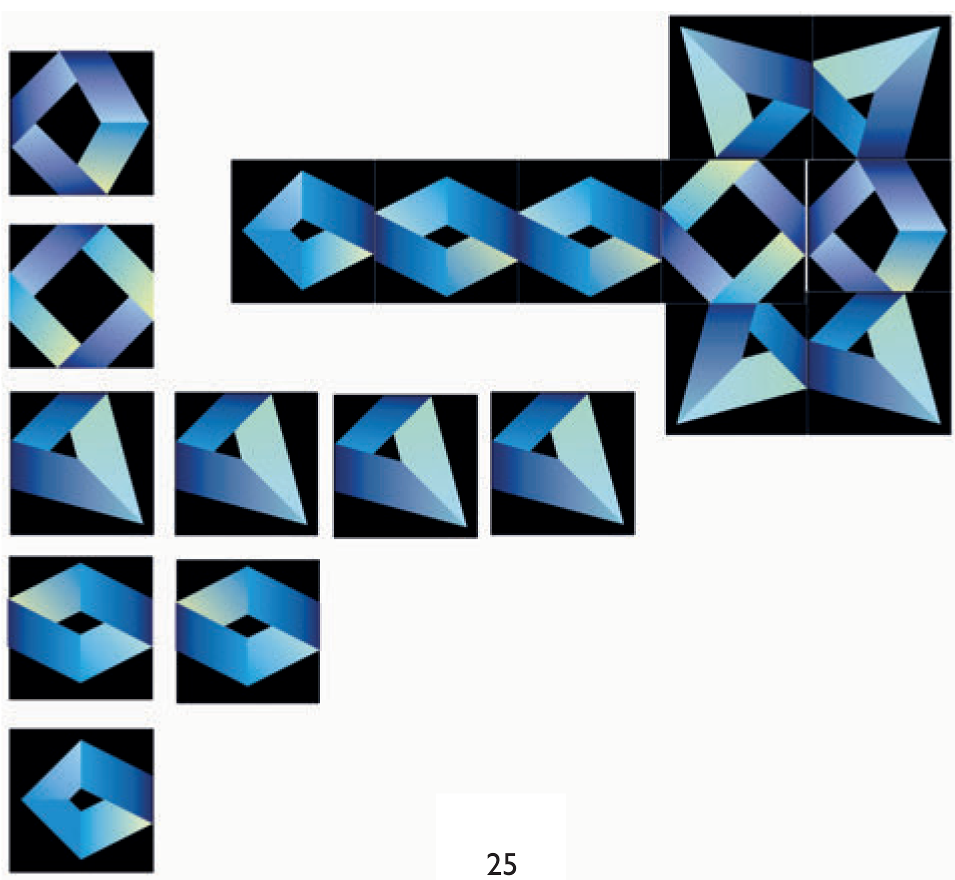


Figure 33

Game 7: Mirror curve and Celtic knot

Experiment with different shapes of mirror curves (Fig 34).

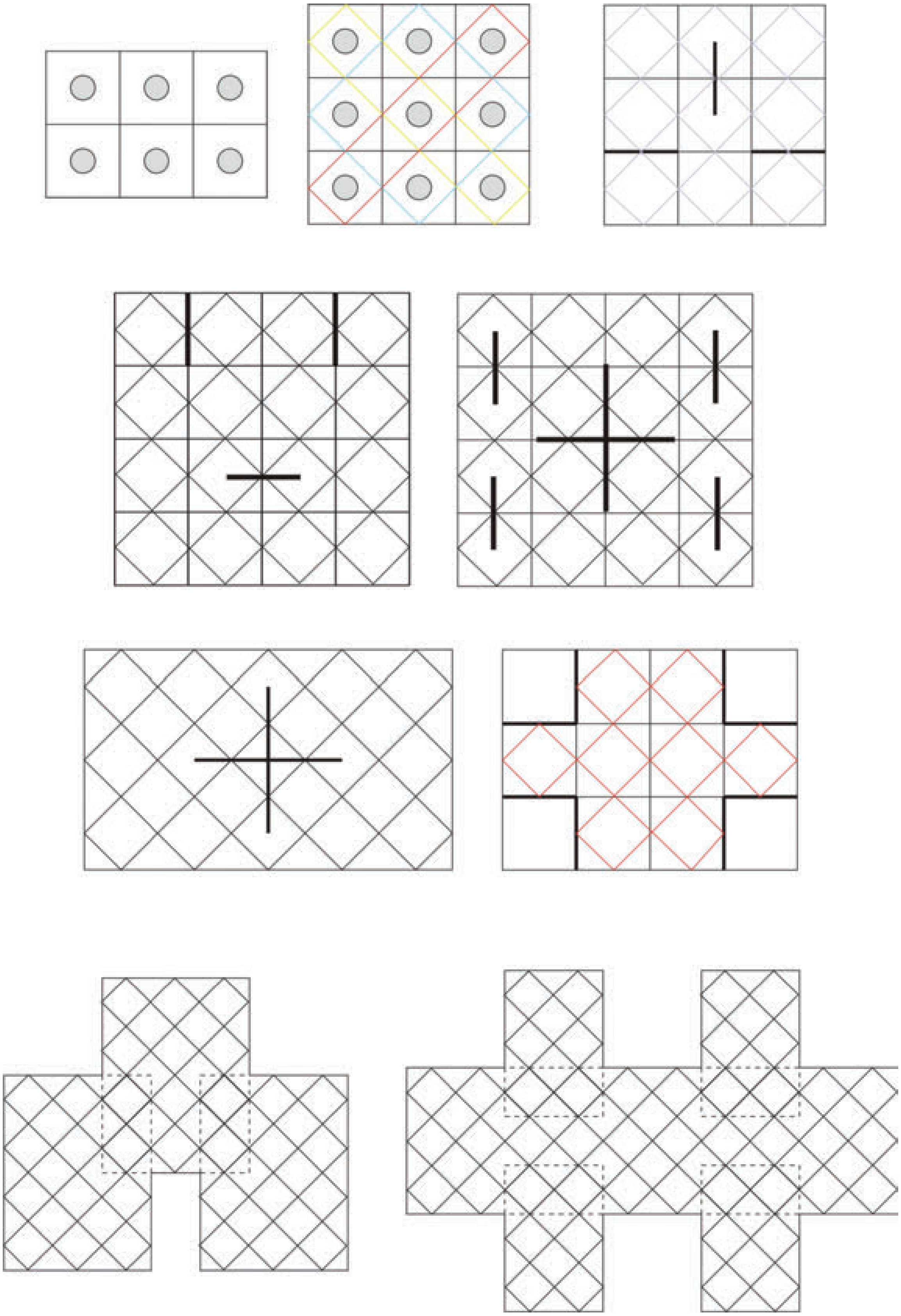
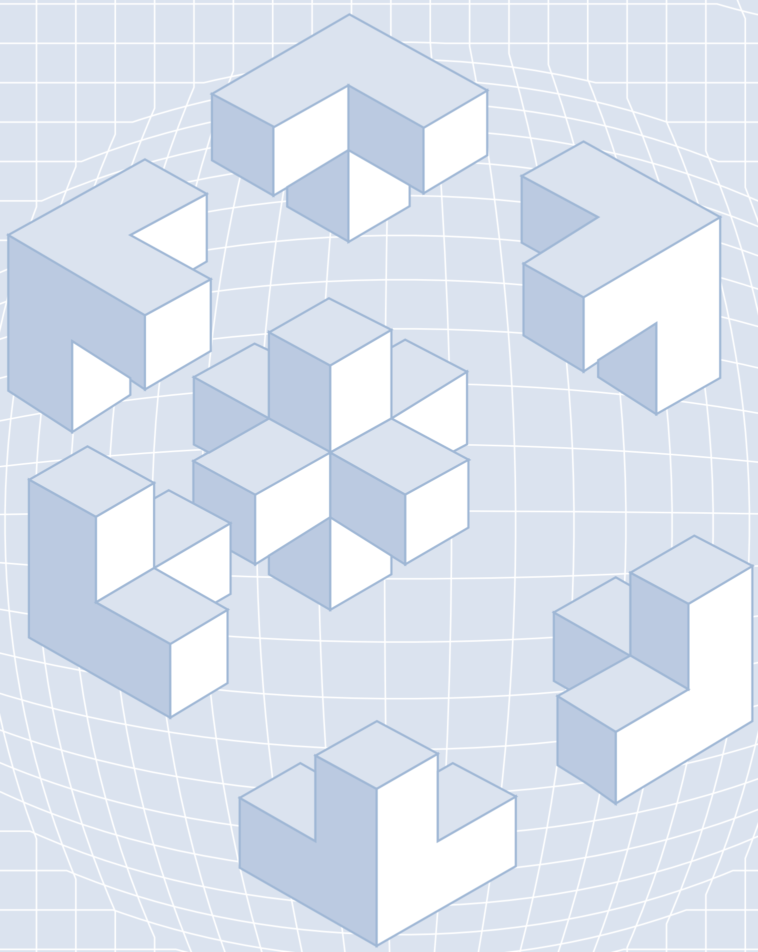


Figure 34

Robert Fathauer

Decoration of Polyhedra using Escher-like Tessellations



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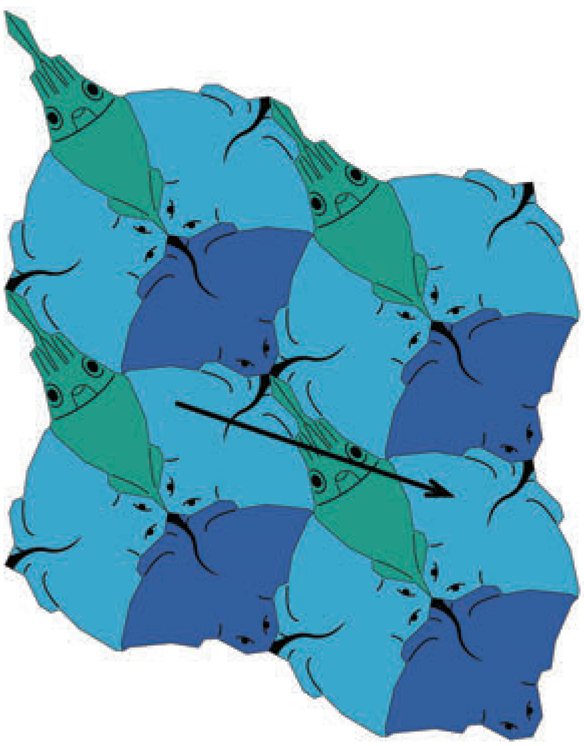
Puzzling Symmetries

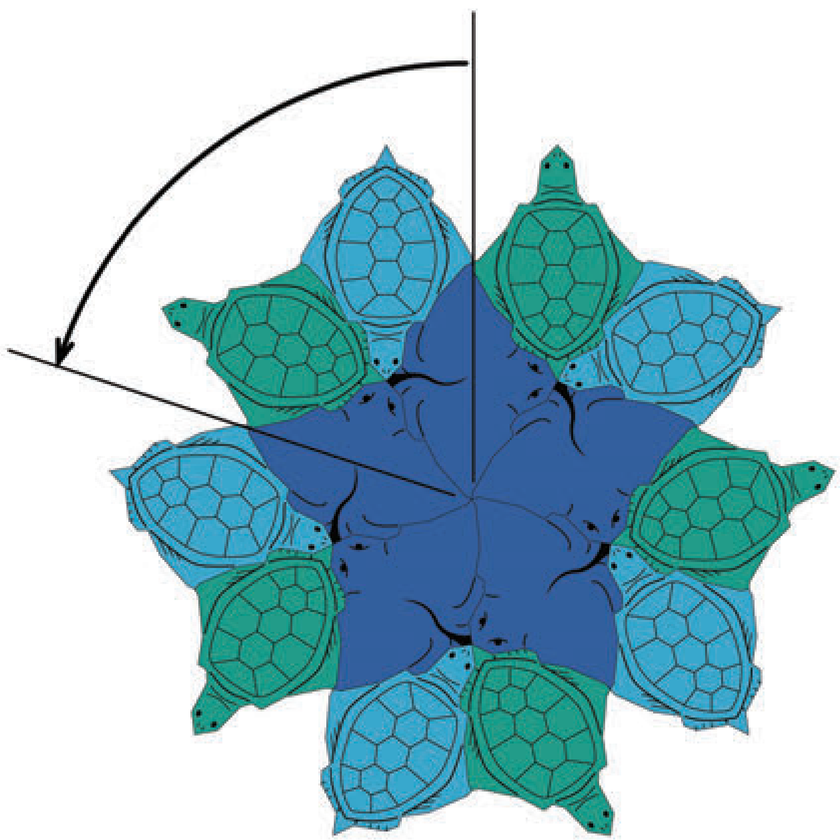
ROBERT FATHAUER is a visual artist, author, and puzzle designer. He has a B.S. degree in Physics and Mathematics from the University of Denver and a Ph.D. in Electrical Engineering from Cornell University. He worked for several years as a research scientist at the Jet Propulsion Laboratory in Pasadena, California. Dr. Fathauer currently owns a small business based in Arizona called Tessellations that produces books, puzzles and other products for mathematics education. His art draws on a range of mathematical ideas, including tessellations, fractals, knots, polyhedra, and hyperbolic geometry. Dr. Fathauer's books include "Designing and Drawing Tessellation" and "Designing and Building Tessellated Polyhedra".

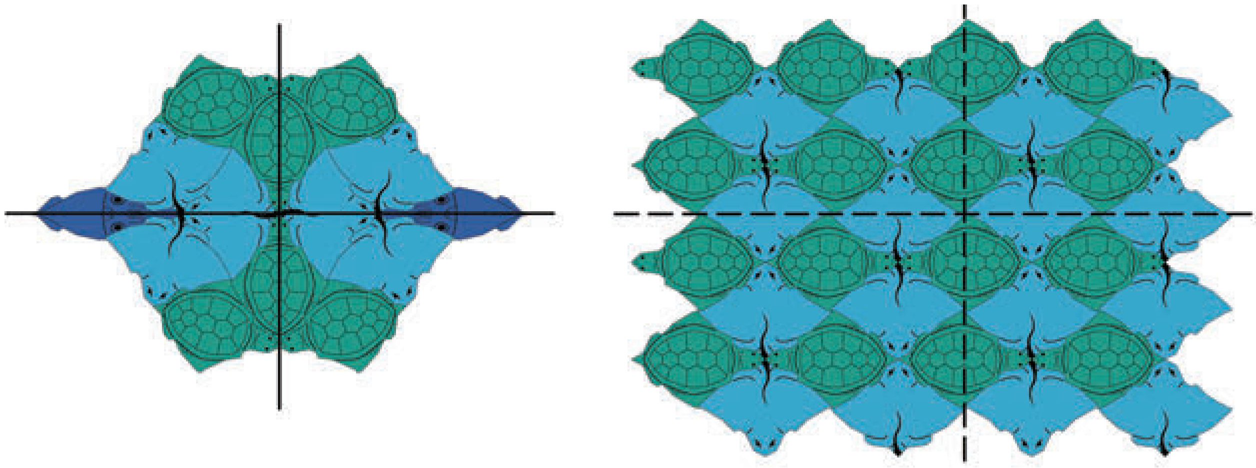
Tessellations

A tessellation is a collection of shapes, called tiles that fit together without gaps or overlaps to cover the infinite mathematical plane. There are three types of symmetry found in tessellations in the plane. A tessellation is said to possess translational symmetry if it can be moved by some amount in some direction and remain unchanged. Such a tessellation is periodic, or repeating. For such a tessellation, an infinite number of different translation arrows can be drawn. In the example at left below, the arrow indicates one such translation.

A tessellation is said to possess rotational symmetry if it can be rotated about a point and remain unchanged. Patterns with both 2-fold and 5-fold rotational symmetry can be constructing using the three types of tiles shown below. This refers to rotation by 1/2 and 1/5 or a full revolution, respectively. In the example at right below, the arrow indicates rotation by 1/5 of a full revolution about the center.

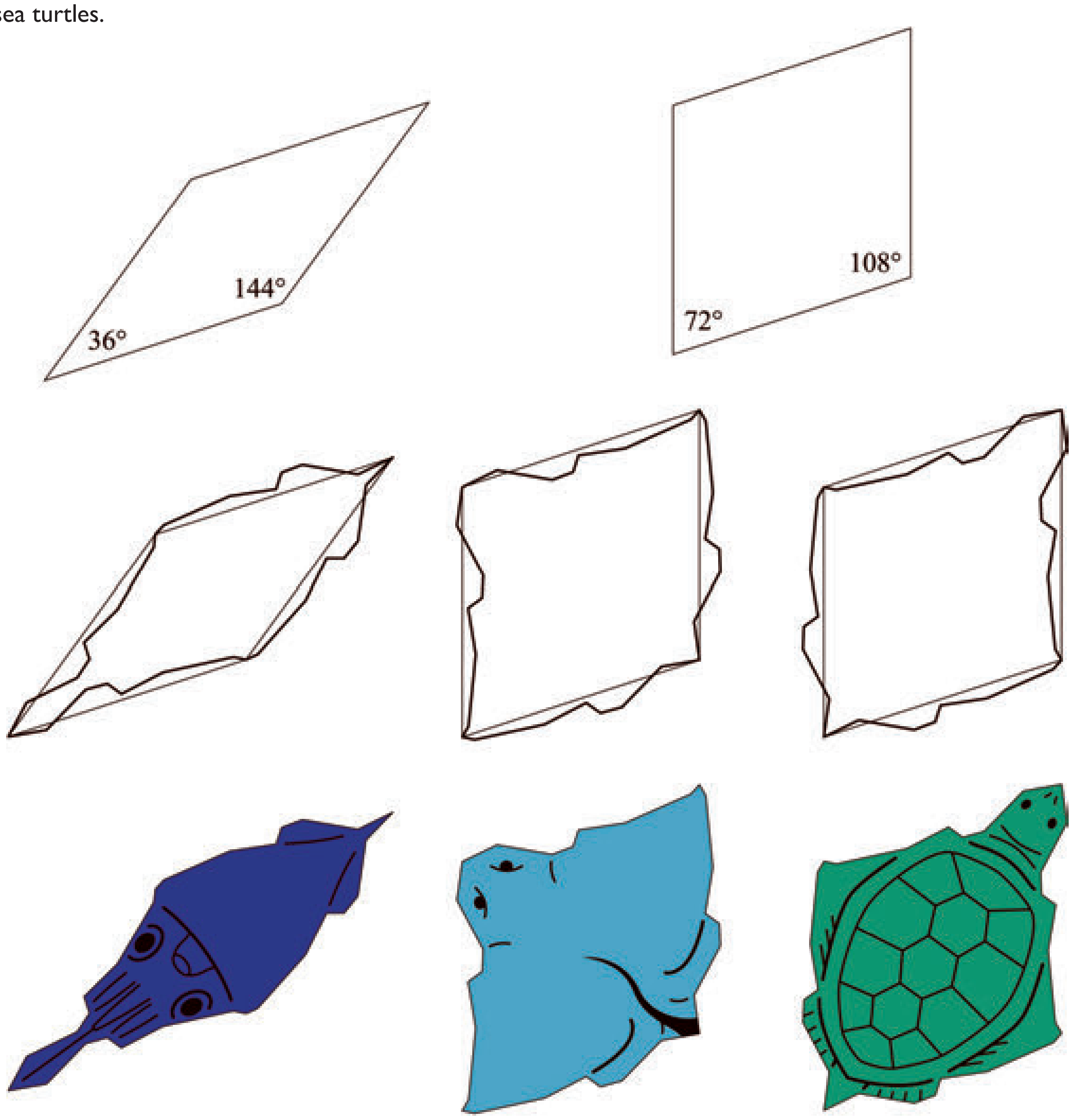






A tessellation is said to possess glide reflection symmetry if it can be moved by some amount along a line and then be reflected about that line and remain unchanged. If the glide distance is zero the tessellation is said to possess mirror or reflection symmetry. In the example at left above, solid lines indicate lines of mirror symmetry. In the example at right above, the two dashed lines indicate two distinct lines of glide reflection symmetry.

The Dutch graphic artist M.C. Escher was famous for creating tessellations in which the individual tiles were recognizable motifs like birds and lizards. An Escheresque tessellation is created by choosing a geometric template and then modifying the edges of the tiles to suggest a natural motif. Interior details are then added to further suggest the motif. In the example below, two different rhombi are modified to form three different types of tiles, squids, rays, and sea turtles.



Activity 1: Symmetry in Plane Tessellations
Teacher's guide

Materials: Copies of Worksheet 1 for students. Colored pencils or markers for coloring in the black-and-white tessellations.

Objective: Become familiar with the three types of symmetry found in plane tessellations and learn how to indicate them.

Vocabulary: Tessellation, translational symmetry, rotational symmetry, and glide reflection symmetry.

Activity Sequence:

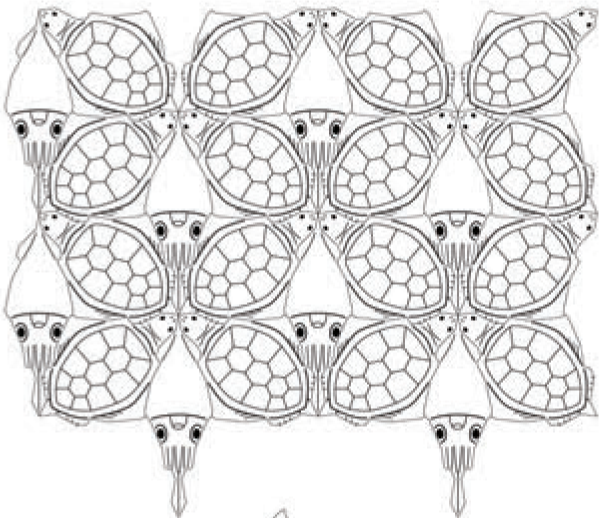
1. Pass out the worksheet. Make sure the students understand that the groups of tiles shown represent portions of tessellations that extend to infinity.

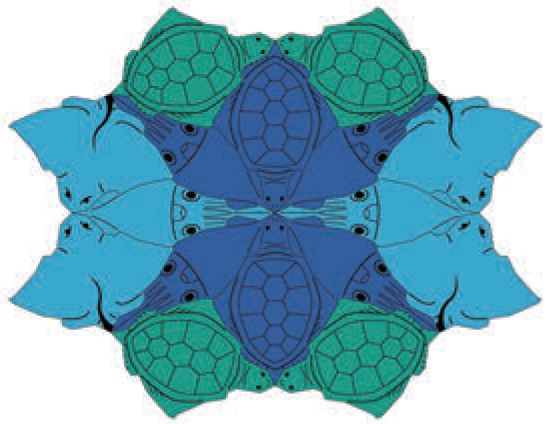
2. Write the vocabulary terms on the board and discuss the meaning of each one.

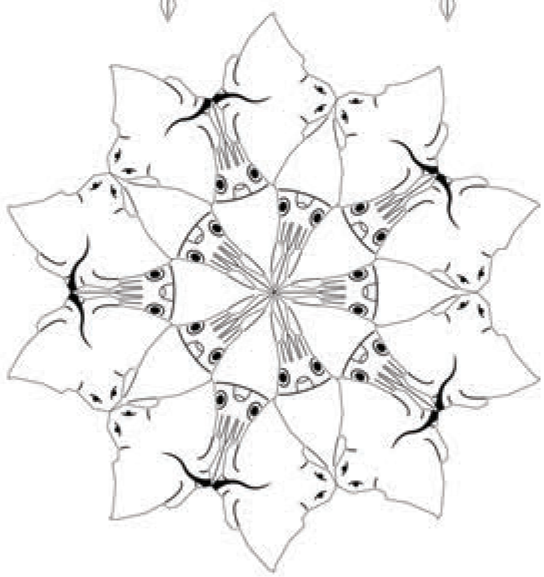
3. Have the students complete the worksheet. Then have students share their answers for each tessellation.
The upper left tessellation has translational symmetry, as well as mirror symmetry and glide reflection symmetry about vertical lines.
The upper right tessellation has mirror symmetry about a single horizontal line and a single vertical line, as well as two-fold rotational symmetry about the center.
The middle left tessellation has 5-fold rotational symmetry about its center.
The middle right tessellation has 2-fold rotational symmetry about its center.
The lower left tessellation has both translational symmetry, and mirror symmetry about horizontal lines.
The lower right tessellation has translational symmetry, and glide reflection symmetry about vertical lines.

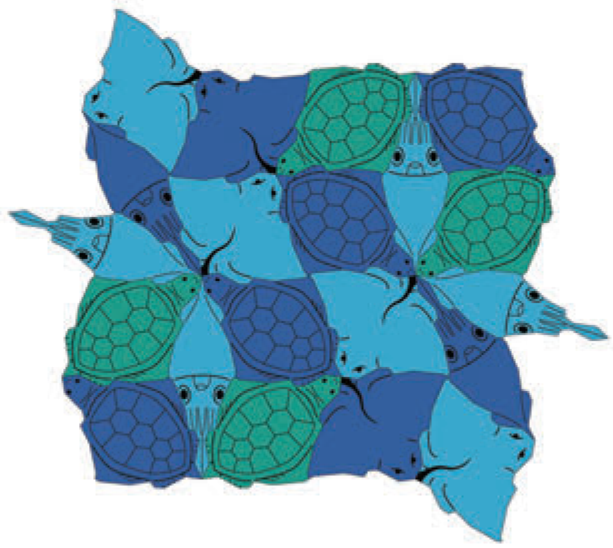
4. The students may color the black-and-white tessellations if they would like.

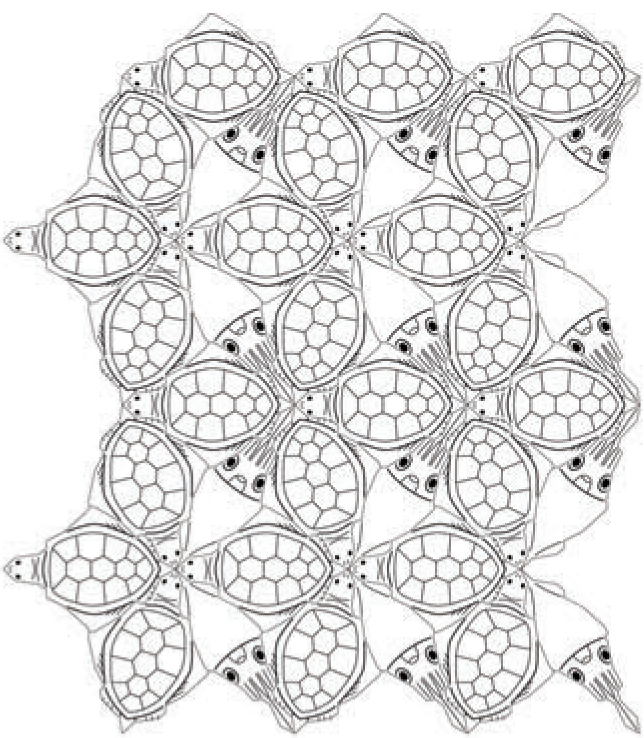
Worksheet 1. Mark each of the tessellations to indicate the symmetry it possesses. Use a "T" to indicate the translation, along with an arrow to indicate the translation distance and direction, a solid line to indicate simple reflection, a dashed line to indicate glide reflection, and a large dot to indicate a center of rotation. For those with rotational symmetry, write a number *n* to indicate *n*-fold symmetry.

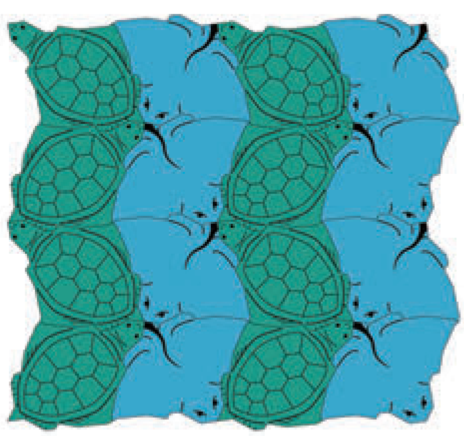








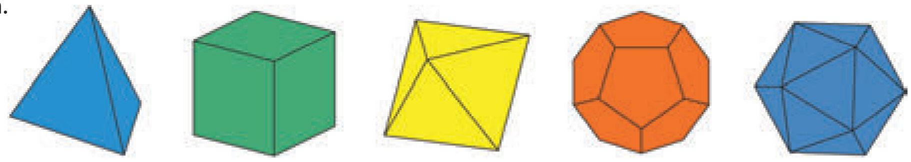




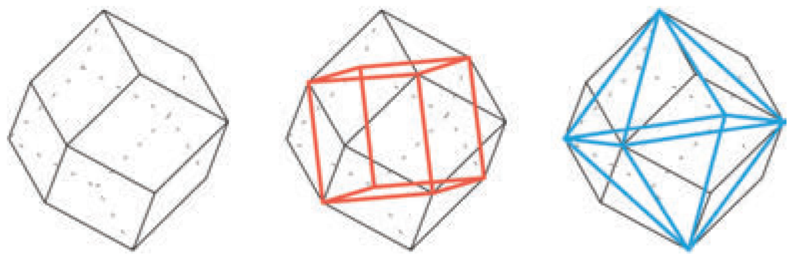
Polyhedra

A polyhedron is a geometric solid, the faces of which are polygons. A face of a polyhedron is thus a flat region of polygonal shape. An edge of a polyhedron is a straight line shared by two adjacent faces. A vertex of a polyhedron is a point where three or more faces meet. A convex polyhedron is one for which any line connecting two points on the surface lies in the interior or on the surface of the polyhedron. Euler's polyhedron formula gives a simple relationship between the number of vertices V, edges E, and faces F for a convex polyhedron: $V - E + F = 2$.

The five Platonic solids are the only convex polyhedra for which each face is the same regular polygon and all vertices are of the same type. From left to right, their names are tetrahedron, cube (hexahedron), octahedron, dodecahedron, and icosahedron.



A regular dodecahedron has 12 faces that are regular pentagons. Another type of dodecahedron is the rhombic dodecahedron, in which each face is a rhombus. The rhombic dodecahedron is closely related to the cube and octahedron. Lines along the short diagonals of the rhombi delineate a cube, while lines along the long diagonals delineate an octahedron, as shown below.



The ratio of the length of a long diagonal to a short diagonal is $\sqrt{2}$, making the angles in the rhombi approximately 70.53° and 109.47°. These are only slightly different than the angles of 72° and 108° used in the squid and sea turtle tiles above. For the last worksheet, the squid and sea turtle tiles were distorted to make them fit the rhombi of a rhombic dodecahedron.

Activity 2: Attributes of the Platonic Solids and Euler's Formula Teacher's Guide

Materials: Copies of Worksheet 2 for students.

Objective: Become familiar with basic attributes of the Platonic solids, such as number of faces, edges, and vertices possessed by each, and become familiar with and learn to apply Euler's formula, in different arrangements, relating these quantities.

Vocabulary: Platonic solid, face, vertex, edge

Activity Sequence:

1. Pass out the worksheet.

2. Write the vocabulary terms on the board and discuss the meaning of each one.

3. Have the students complete the table at the top of the worksheet. Then have students share their answers and fill in the table on the overhead copy of the worksheet.

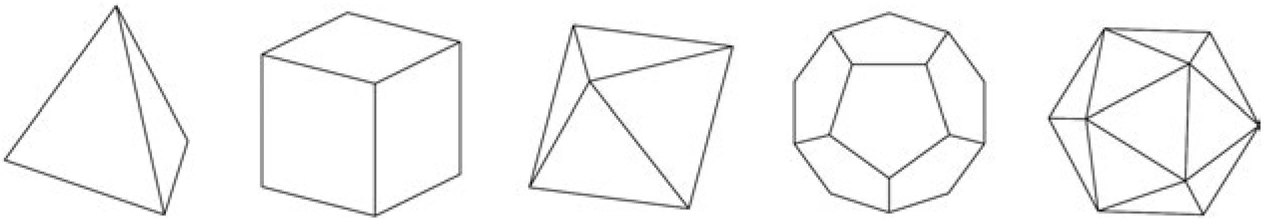
Tetrahedron	*4*	*6*	*4*	*3*
Cube	*6*	*12*	*8*	*3*
Octahedron	*8*	*12*	*6*	*4*
Dodecahedron	*12*	*30*	*20*	*3*
Icosahedron	*20*	*30*	*12*	*5*

4. Have the students complete the next three tasks on ordering and grouping the solids. Then have students share their results and write the results on the overhead copy of the worksheet.

5. Discuss Euler's formula. Ask the students to calculate the value of n for each of the five Platonic solids. Be sure they all understand why it is 2, and then have them solve the remaining problems on the worksheets. Have the students share their answers.

14, 6, 36.

Worksheet 2. Attributes of the Platonic Solids and Euler's Formula



Fill out the table below for the properties of each of the Platonic solids.

Name of the solid	*Number of faces F*	*Number of edges E*	*Number of vertices V*	*Number of faces meeting at each vertex*

List the Platonic Solids by number of faces, from least to most.

Group the Platonic Solids by the number of edges possessed by each polygonal face, from least to most. How many groups are there?

Group the Platonic Solids by the number of faces meeting at each vertex, from least to most. How many groups are there?

Euler's formula provides a simple relationship for the number of vertices, faces, and edges in a convex polyhedron: $V + F - E = n$. Using the table above, determine the value of n. Then use the formula to answer the questions below.
A cuboctahedron has 12 vertices and 24 edges. How many faces does it have?

A truncated octahedron has 24 vertices, 36 edges, and 8 hexagon faces. The remaining faces are square; how many square faces are there?

A truncated cube has the same number of faces as a truncated octahedron, and 24 vertices. How many edges does it have?

Activity 3: Rhombic Dodecahedron Decorated with a Tessellation
Teacher's guide

Materials: Copies of Worksheets 3a and 3b for students. Colored pencils or markers, scissors, and glue or tape. If possible, use a heavy paper such as card stock for the copies, as the polyhedra will be sturdier.

Objective: Understand the relationship between the rhombic dodecahedron and the cube and octahedron, and learn how a plane tessellation can be adapted to tile a polyhedron.

Vocabulary: Rhombic dodecahedron.

Activity Sequence:

1. Pass out Worksheet 3a. Three copies are needed to form one polyhedron.

2. Write the vocabulary term on the board and discuss its meaning.

3. Have the students draw the diagonals of each rhombus on Worksheet 3a using colored pencils or markers.

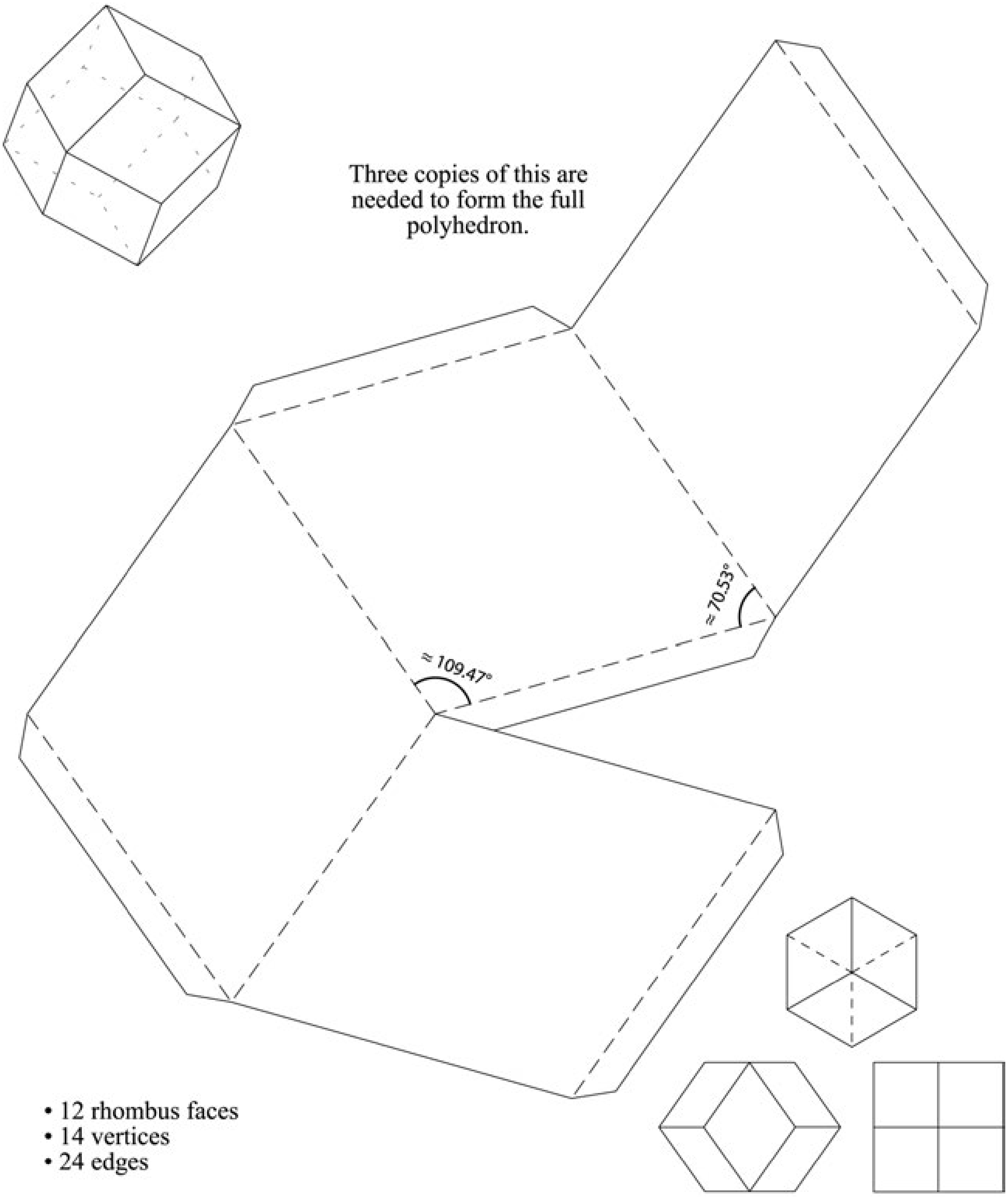
4. Have the students construct the polyhedron on Worksheet 3a.

5. Ask students to describe how one set of diagonals shows the edges of a cube, and the other shows the edges of an octahedron.

6. Have the students construct the polyhedron on Worksheet 3b.

7. If there is time, students can draw their own decorative design on copies of Worksheet 3a and construct a polyhedron with their own decoration.

Worksheet 3a. Use a colored pencil or marker to draw the long diagonal of each rhombus. Then use a second color to draw the short diagonals. Finally, build the rhombic dodecahedron, by cutting along the solid lines and then folding along the dashed lines. The tabs go under the faces, and tape or glue can be used to hold it together.

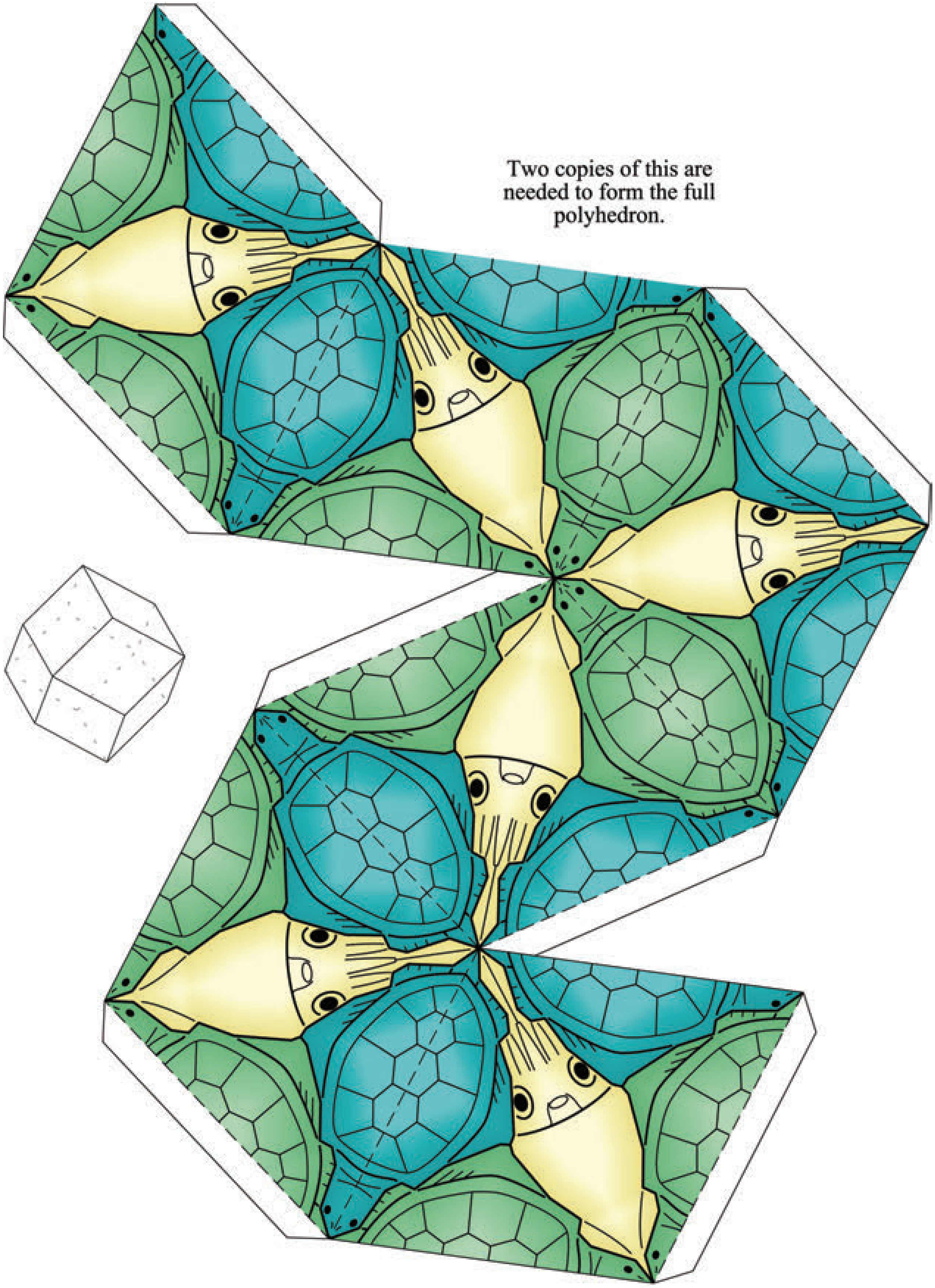


Three copies of this are needed to form the full polyhedron.

- 12 rhombus faces
- 14 vertices
- 24 edges

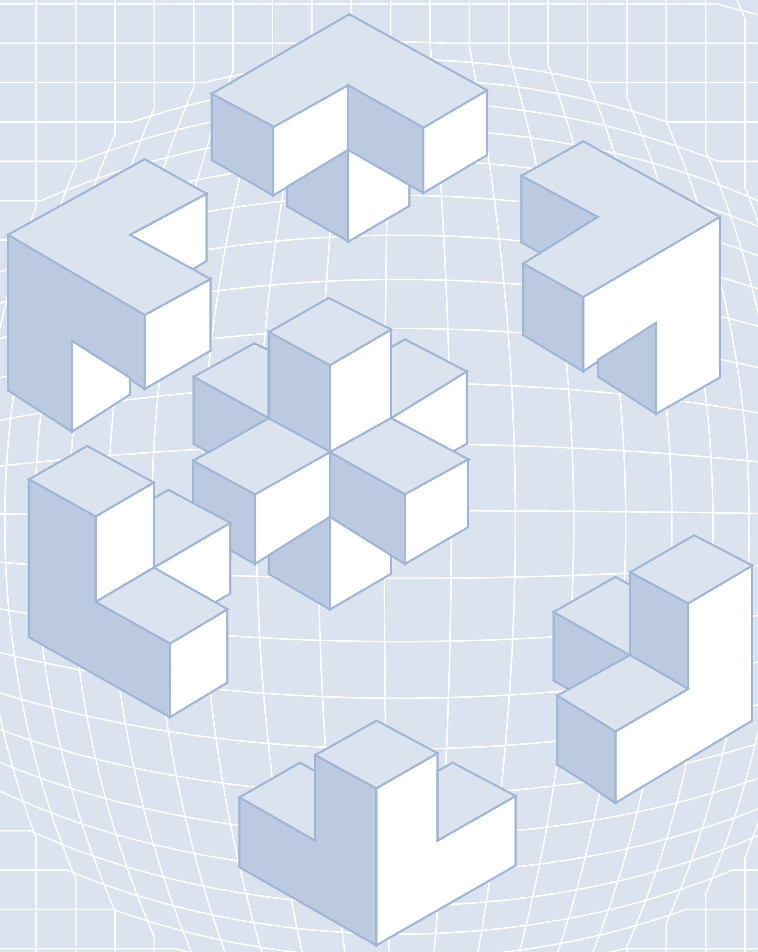
HIgh-symmetry views

Worksheet 3b. Construct a rhombic dodecahedron decorated with squid and sea turtle motifs.



Ilona Oláhné Téglási

Tangram-type Games and Matchstick Puzzles



Visuality & Mathematics

Experiential Education of Mathematics through Visual Arts, Sciences and Playful Activities

Puzzling Symmetries

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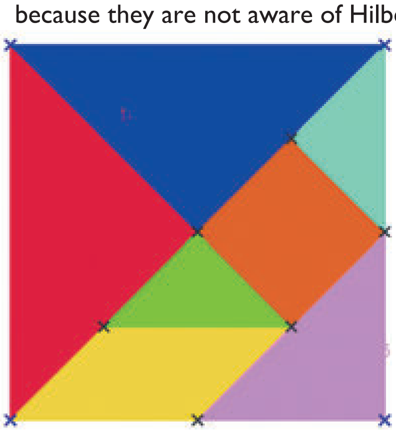
1. Tangram-type games developing the concept of area and volume

General introduction: When teaching the area of polygons, teachers often recognize that pupils just blindly learn the formula of the area of different polygons. However, there's no real understanding behind the formula. This can be remedied with the ancient Chinese game "Tangram" which requires players to make different figures from a square divided into 7 pieces. This can help pupils to develop a deeper understanding of the concept of area. The objective of the puzzle is to form a figure (given only in silhouette) using all seven pieces, which may not overlap. It was originally invented in China, and then brought over to Europe by trading ships in the early 19th century. It became very popular in Europe for a time. During World War I it was one of the most popular dissection puzzles in the world. Over 6500 different tangram problems have been compiled from 19th century texts alone, and the number is still rising – the number of possible figures is, however, finite. Fu Traing Wang and Chuan-Chin Hsiung proved in 1942 that there are only thirteen convex tangram configurations.[5]

One important property of the area of a plane figure – according to Hilbert's axiom – is that if we divide a plane figure into pieces, the sum of the area of the pieces is equal to the area of the original figure. We often use this property when we have to measure the area of a complex figure: we divide it into simple polygons we can measure easily. In such problems we find, that pupils, who have only learned the formula, cannot even begin to solve them, because they are not aware of Hilbert's axiom.



It is very helpful to play Tangram when students are first introduced to the concept of area (5–6 grade), so that pupils will be able to use this method later in secondary school tasks. We can use the original game, as there are many websites on the Internet from where we can get the figures [4, 5, 6]. (The original dissection you can see in Fig.1.)

From the thousands of possible shapes we should select those that best fit to the requirements of the mathematics curriculum. Or we can make new forms, such as in the following problem.

Figure 1. Original Chinese Tangram square

Problem: From this square – divided into 5 pieces – form a rectangle, a parallelogram, a triangle and a cross by putting all the elements together in another way, without gaps or overlapping!(originally devised by Sam Loyd)[3, 4, 6]

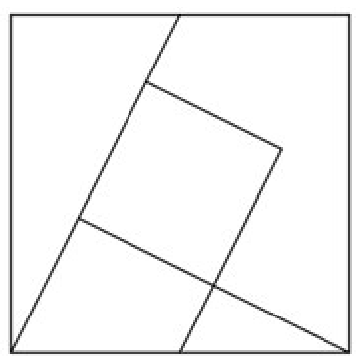


Figure 2. Sam Loyd's tangram

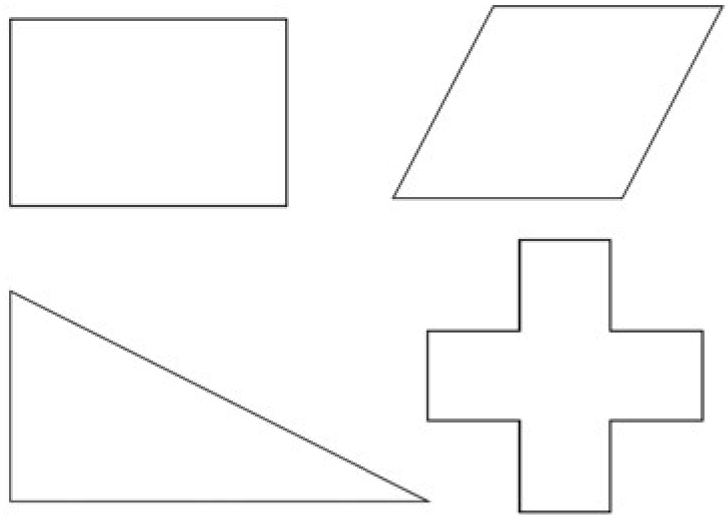


Figure 3. Conversed figures

Solutions:

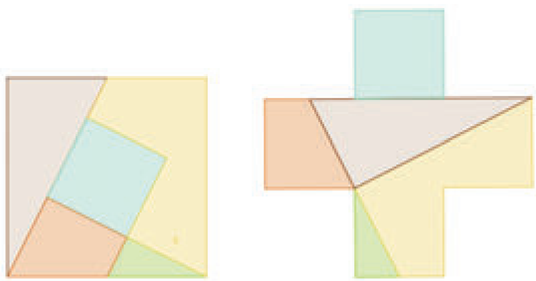


Figure 4. How to form the cross

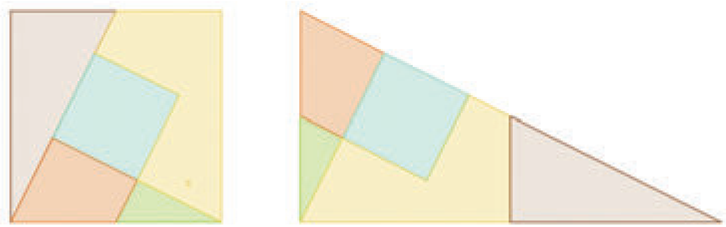


Figure 5. How to form the triangle

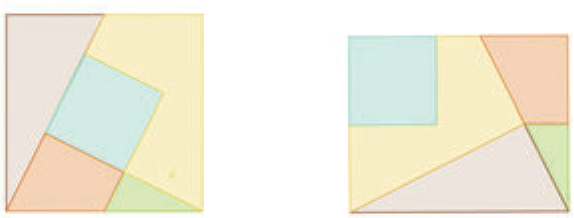


Figure 6. How to form the rectangle

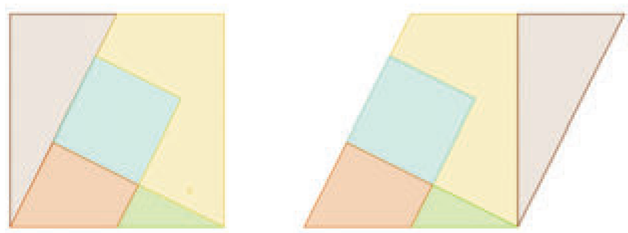


Figure 7. How to form the parallelogram

After creating the different polygons, we can conclude with our students, that the area of these four different polygons is equal to the area of the original square: $T = a^2$. For lower grades we can discuss other properties of these polygons too. We can also produce other problems based on these figures (depending on the grade), for example [2]:

1.1 Using the original square's area, determine the area of each of the 5 pieces!

Answer: the blue square's area is $\frac{1}{5}T$ the "big" triangle's area is $\frac{1}{4}T$ the "small" triangle's area is $\frac{1}{20}T$ the trapezoid's area is $\frac{3}{20}T$ the concave hexagon's area is $\frac{7}{20}T$.

1.2 What are the lengths of the sides of the 5 pieces when making up the original square's side? How many different lengths are there?

Answer: We can discuss with the pupils that E, G and F points are midpoints of the sides of the square. If we draw in some auxiliary segments, we can see, that there are four different lengths we have to determine: $\overline{BE}$, $\overline{GH}$, $\overline{HI}$, $\overline{ED}$. We can easily find these lengths by using the Pythagorean theorem and similarity.(Fig. 8.)

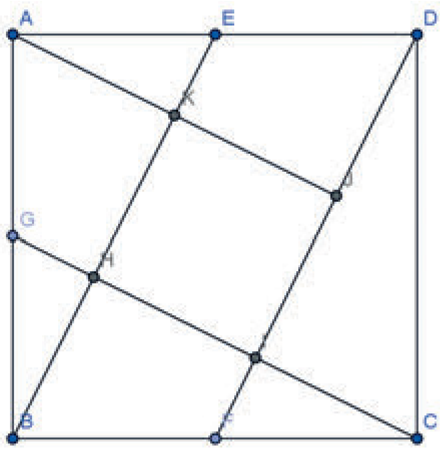


Figure 8: Construction view of the puzzle

Because $a^2+\left(\frac{a}{2}\right)^2=\overline{BE}^2$, $\overline{BE}=\frac{a\cdot\sqrt{5}}{2}$. $ABE\Delta\sim HBG\Delta$. (both are rectangular, and have a common angle at vertex B), so $\frac{\overline{GH}}{\overline{GB}}=\frac{\overline{AE}}{\overline{EB}}$, and $\overline{GB}=\overline{AE}=\frac{a}{2}$ so thus $\overline{GH}=\frac{a\cdot\sqrt{5}}{10}$. Because of the similarity of these two triangles, we can also conclude that $\frac{\overline{HB}}{\overline{GH}}=\frac{\overline{AB}}{\overline{AE}}=\frac{2}{1}$, so $\overline{HB}=\frac{a\cdot\sqrt{5}}{5}$, and $\overline{HB}=\overline{HI}=\overline{IJ}$ .. And we know, that $\overline{ED}=\frac{a}{2}$ (Fig. 8.).

1.3 What are the peripheries of the 5 pieces?

Answer:

$P_{ABE}=a+\frac{a}{2}+\frac{a\cdot\sqrt{5}}{2}=\frac{a(3+\sqrt{5})}{2}$, $P_{BFIH}=\frac{a}{2}+\frac{a\cdot\sqrt{5}}{10}+2\cdot\frac{a\cdot\sqrt{5}}{5}=\frac{a\cdot(1+\sqrt{5})}{2}$, $P_{FCI}=\frac{a}{2}+\frac{a\cdot\sqrt{5}}{5}+\frac{a\cdot\sqrt{5}}{10}=\frac{a(5+3\sqrt{5})}{10}$, $P_{HIJK}=4\cdot\frac{a\cdot\sqrt{5}}{5}$, $P_{CDEKJI}=a+\frac{a}{2}+\frac{a\cdot\sqrt{5}}{10}+3\cdot\frac{a\cdot\sqrt{5}}{5}=\frac{a(15+7\sqrt{5})}{10}$.

1.4 What are the peripheries of the conversed figures?

Answer: We have already discussed, that the area is the same, and now we can see, that we could make figures with the same area and a different periphery. The periphery of the cross (Fig.4) is $12\cdot\frac{a\cdot\sqrt{5}}{5}$, the periphery of the triangle (Fig.5) is $3a+2\cdot\frac{a\cdot\sqrt{5}}{2}=a(3+\sqrt{5})$, the periphery of the rectangle (Fig.6) is $2\cdot\left(\frac{a\cdot\sqrt{5}}{2}+2\cdot\frac{a\cdot\sqrt{5}}{5}\right)=\frac{a\cdot 9\sqrt{5}}{5}$ and the periphery of the parallelogram (Fig.7) is $2\left(a+\frac{a\cdot\sqrt{5}}{2}\right)=a(2+\sqrt{5})$.

With this simple dissection of a square we can motivate pupils, and by solving this series of exercises we can develop their concept of area and periphery.

With 3D dissection puzzles we can reach a similar development in the concept of volume. We can also use 3D dissection puzzles to help the students understand the properties and volume of polyhedra. There are different polyhedra you can obtain: cubes, tetrahedron, pyramid, 3D cross, "star", etc [8].

For pupils it is often difficult to understand why the volume of a 3-unit long sided cube is equal to that 27 volume units. Maybe, because they cannot imagine the inside of the cube. If they take the "cubic snake" (Fig.9-10) apart, they can physically count the number of volume units in the cube.

Figure 9. Cubic snake packed

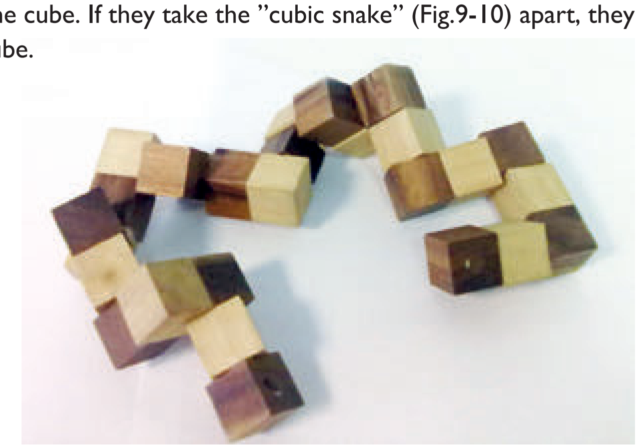


Figure 10. Cubic snake opened up

Another example is the following puzzle cube, which can be taken apart to three cubic units and four cubic units (Fig.11-12). One task is to put the parts together, to form the cube, and another to measure the volume. Through these games we can combine playing with serious learning – and thus the pupils will be motivated to do the tasks.

Figure 11. 3D puzzle cube packed

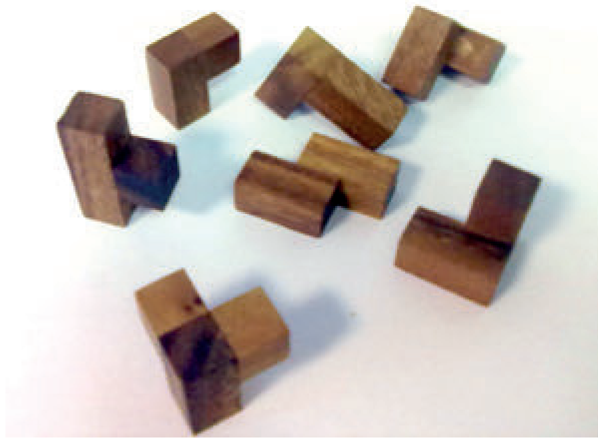


Figure 12.3D puzzle cube taken apart

At first sight they are only good games – but on reflection they also make our children think things through, motivate them to learn, and show them how to problemsolve. For a teacher of mathematics these games give so much more: they are good tools for developing mathematical skills such as: spatial ability, logical thinking, analogical and deductive thinking and, learning problem solving strategies.[2]

In short, by using the Chinese game "Tangram" in the classroom we can create a series of problems which can be interesting, motivating and also fit into the mathematics curriculum.

2 Matchstick puzzles in mathematics lessons

General introduction: Most of us can recall matchstick party games, where we have to remove or replace matchsticks in a given figure to form another figure. Everyone likes such puzzles, they are useful for developing teamwork and need only a few boxes of matchsticks. There are many different types of this puzzle, one can create one's own or use the Internet to locate those, that fit to the topic of a particular mathematics lesson. These puzzles can develop logical thinking skills, recognition of spatial connections, properties of plane figures, geometrical transformations, measuring length and area – just to mention a few.
We can divide these puzzles into different groups, according to the task we have to solve:

1. **forming other polygons from given polygons, using the properties of the polygons;**
2. **forming different polygons from a given number of matchsticks;**
3. **logical problems – visualization of non-geometric problems**.[1]

In the present book we can show only a few of the possible problems, and make some methodological remarks on them. We can find many of these puzzles on the Internet [7].

Problems:

1. In these problems we use the matchsticks as length units, and we play with the different places and size of the polygons, as in the following examples. These exercises are good for younger age groups, when introducing the different polygons and transformations.
   1.1 Move 2 sticks to get 3 small and 1 big square!

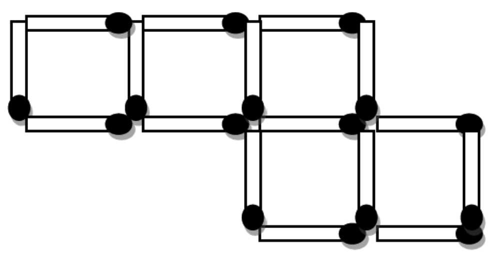


Figure 13.3+1 squares puzzle

1.2 Move 2 sticks to get 4 regular triangles!

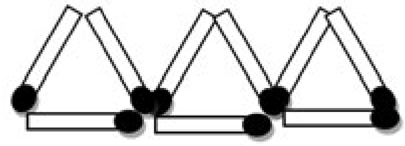


Figure 14. Four triangles puzzle

1.3 Move 4 sticks to get 6 regular triangles!

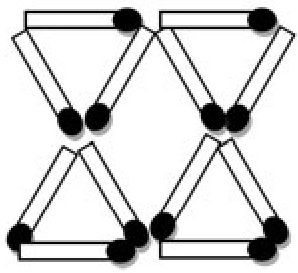


Figure 15. Six triangles puzzle

Solutions:

1.1 The key is, that the "big square" has sides measuring 2 units each, thus we use translation, so the transformed figure is:

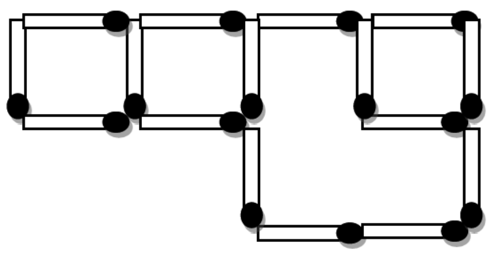


Figure 16. Solution of 3+1 square puzzle

1.2 The key is that we use a central symmetry to get 4 triangles:

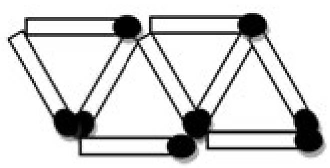


Figure 17. Solution of four triangles puzzle

1.3 In this case we use translation when we move the matchsticks:

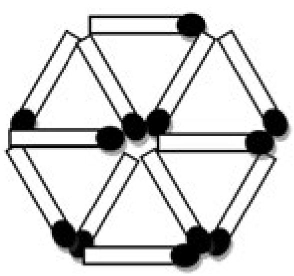


Figure 18. Solution of six triangles puzzle

2. The next problems can help the pupils understand the difference between area and periphery, and can be used to introduce the "isoperimetric problem" in secondary school. Again we use the matchstick as length unit, and the area of a square formed by 4 matchsticks as area unit.
   - 2.1 Form as many different simple polygons as you can using 12 matchsticks, they cannot cross each other (every matchstick has to be on the periphery of the polygon)!
   - 2.2 Form convex and concave polygons from 12 matchsticks!
   - 2.3 What is common to all formed polygons?
   - 2.4 Try to give the area of each polygon you've formed (for secondary school students)!
   - 2.5 Which polygon has the biggest area, and which the smallest?

Solutions:

2.1 Pupils can form many different polygons – ask them to define, which polygon they've formed (triangle, rectangle, hexagon, etc.). Maybe, you could explain what a "simple polygon" means. The number of formed polygons can give us a view of the pupils' creativity (they can also work in groups) [2]. Maybe, all the polygons will be convex – so comes the next task.

2.2 If they didn't form concave polygons in the first task, they will probably now form crosses, different "stars", etc. There are plenty of possible figures, we cannot state a number, it all depends on the time given to the task and also the pupils' creativity.

2.3 What these polygons have in common, is that the periphery is the same, 12 units long. We can discuss with the pupils, that the same periphery can encompass different areas ("dual" problem to the square dissection problem above).

2.4 This task is mainly for secondary school students, because in some cases trigonometric formulae need to be applied. This exercise is also useful for discussing the original meaning of measurement – we give the length unit and the area unit, use the axioms of measurement – and the length of the matchstick as unit (don't use centimetres, because it detracts from the uniqueness of this exercise!). We can also ask the students to put the polygons into a sequence of increasing or decreasing area.

2.5 After giving the area of some formed figures, the students can draft a hypothesis: that the regular 12-sided polygon has the biggest area– even at this stage we are encountering the isoperimetric problem. It depends on the group and students' age, how deeply we get into discussing this problem. On the other hand, which polygon has got the smallest area, is another interesting question: try to ask the students leading questions that take them to the hypothesis: we can give a smaller area from every given positive number. For example, forming a parallelogram, and decreasing angles of the sides from 90° to 0° (Fig.19.). To prove this thesis we need analytical tools, so this part of the problem also depends on the group we are working with.

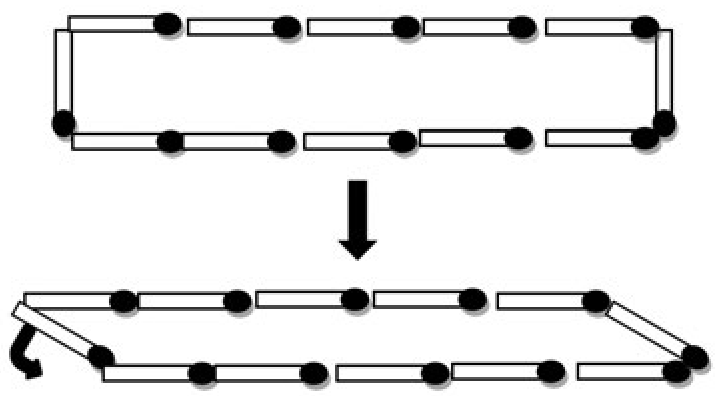


Figure 19. Decreasing the area and keeping the periphery

3. The next problems are for developing logical skills, require creativity and can be used as motivation before a more serious mathematical problem with matchsticks, like those above, or perhaps as a wind down at the end of a lesson. But before we think that these are just for fun, we should mention that the solutions to these problems exercise students' thinking skills (such as analytical-synthetical thinking, changing view, recognizing spatial connections, forward and backward thinking, even cultural and linguistic skills, etc.).[1,2]

Problems:

3.1 Make the equation true by moving only 1 matchstick!

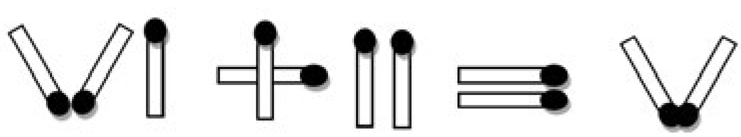


Figure 20. Equation puzzle

3.2 Arrange 6 matchsticks together, so that each stick touches every other stick!

3.3 This is a running horse. Move 1 matchstick to make the horse move in another direction!

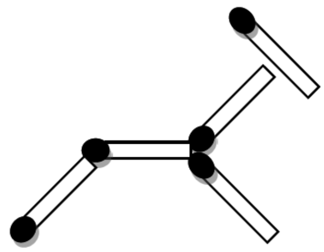


Figure 21. Horse puzzle

3.4 How could we make this equation true without adding, taking away or moving any matchsticks?

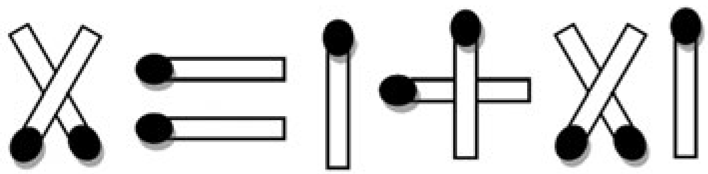


Figure 22. Tricky equation puzzle

I won't give the solutions to the last four problems – it is a challenge for the teachers too. You can exercise your own logical thinking skills! I hope everyone will enjoy thinking about them.

Finally I'd like to mention that these are only a small selection of how we can use matchsticks in mathematics lessons. I hope I have given you the inspiration to try them, and to create your own matchstick puzzles for your classes!

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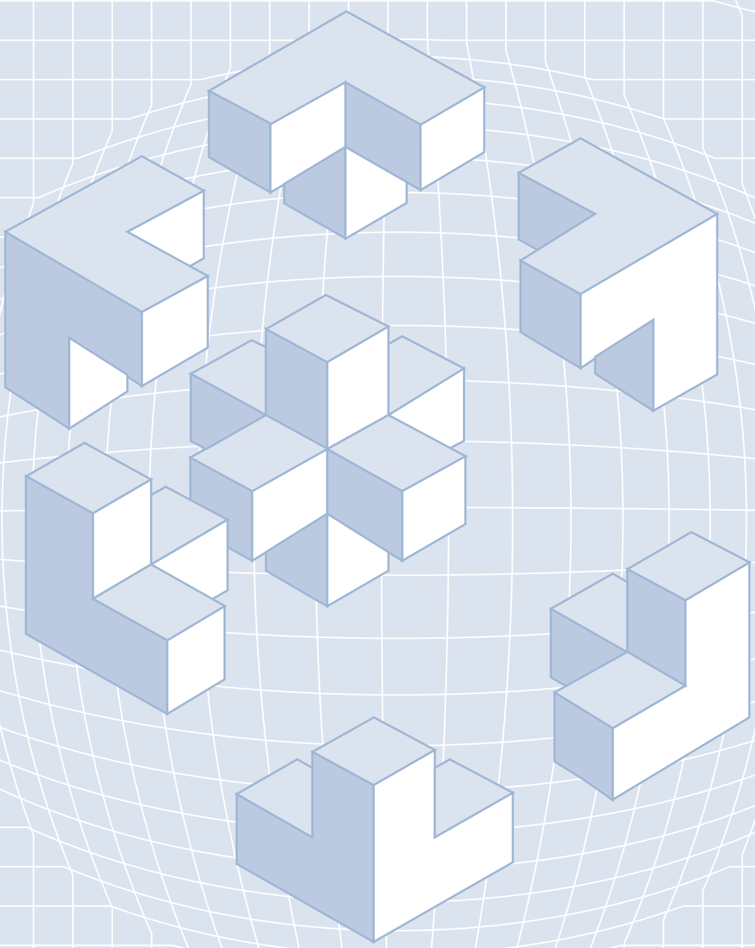
6. Sam Loyd's book of tangrams (Dover Publications Inc., 2007)
7. http://matchstickpuzzles.blogspot.hu/
8. http://www.theiqgames.com

Eleonóra Stettner

Möbius Strip and Frieze Symmetries



Visuality & Mathematics
Experiential Education of Mathematics through Visual Arts, Sciences and Playful Activities

Puzzling Symmetries

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Introduction

The aim of this paper is to show how many interesting problems and areas of mathematics can be explored from a simple strip of paper. For example, we can examine the sides of a surface or we can create a coloured map on an interesting surface that leads us to a special graph. Graphs can be represented by matrices in many ways. For example if a symmetrical line-pattern is drawn on a paper strip, then the pattern can be made infinite by joining the ends of the strip together. Thus we return to our original question as to whether a surface with one or two sides preserves the given frieze symmetry?

Some Simple Features of the Möbius Strip

The Möbius strip or Möbius band is the best-known surface with only one side or as it is also referred to, non-orientable. What does it mean that a surface has only one side? It can be determined in various ways. If a relatively long rectangle is taken, then the two shorter sides can be joined together in two different ways: as a cylindrical surface or as a Möbius strip by taking a paper strip and giving it a half-twist, and then joining the ends of the strip together to form a loop (Figure 1). The entire length of the Möbius strip (on both sides of the original paper) can be painted with a wide paintbrush without ever crossing an edge. It does not work on a cylindrical surface that has two sides, that is orientable. The Möbius band has one surface not like the cylindrical surface with an inside and outside surface.

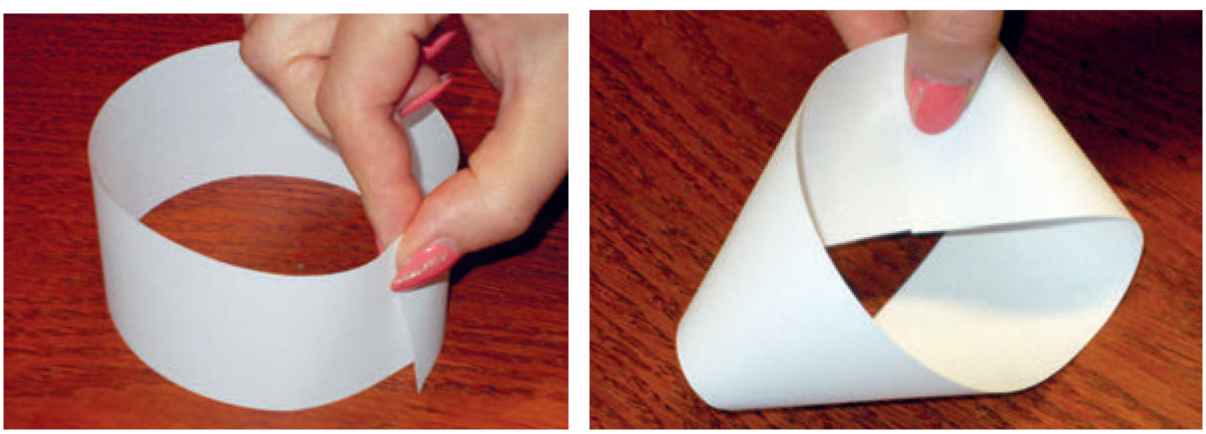


Figure 1: a) A cylindrical surface from a paper strip   b) A Möbius strip from paper

An interesting result is gained if we cut out the second strip in Figure 13 joining the two shorter ends together as a Möbius band. Then we cut the surface along the broken line (along the midline). What dowe get in this way? How many times is the band twisted? Does the resulting surface have one or two sides? This latter feature can be decided on by drawing a line approximately parallel to the edge of the strip along the midline from a point on the surface and if the line ending up in the starting point was drawn along the length of the entire strip then the surface has one side, however, if there is any part of the surface where the midline was not drawn then the surface has two sides. We gain a surprising result even if we cut out the third strip in Figure 12, giving it a half-twist, and then joining the ends of the strip together to form a loop and cutting it along the trisecting line of the surface.

Möbius Strip and Map Colouring

A classic problem with a long history is the so-called "four colour problem" or the "four colour map theorem" stating that, on a plane or on a sphere no more than four colours are required to colour the regions of the map so that no two adjacent regions have the same colour. Two regions are called *adjacent* if they share a common boundary that is not a corner, where corners are the points shared by three or more regions. The theorem was proven in 1976 by Kenneth Appel and Wolfgang Haken, by reducing the infinite number of potential maps into particular sets of maps and then by using a special-purpose computer program to confirm that each of these maps had this property. It is interesting and surprising that the problem of map colouring was solved on a Möbius strip earlier than in the case of planar maps. How many colours are required to colour any map on a Möbius strip so that no two adjacent regions have the same colour? On a Möbius strip this number is six, see Figure 2. Cut out the coloured paper strip in Figure 13, make a Möbius strip and count how many neighbours each region has. Is it five?

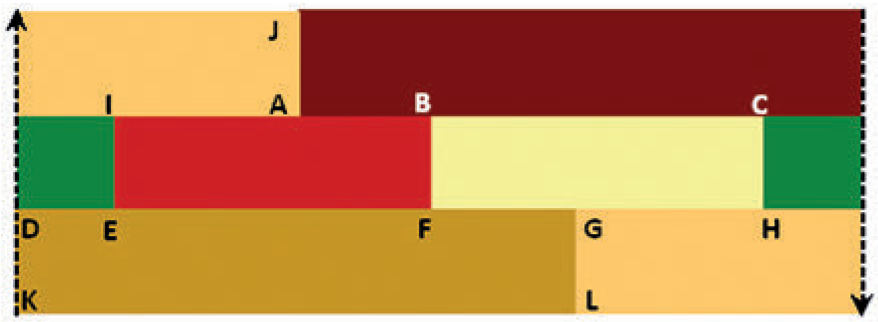


Figure 2: "Map" on the Möbius strip, each "country" has five neighbours

The Tietze Graph

Map colouring of the Möbius strip leads to some problems for graph theory. Regions on this map are rectangles (it is certainly not necessary, regions can be of other shapes). Label the vertices of the rectangles as shown in Figure 2. In this way a graph is obtained. Vertices of the graph are the labelled points, its edges are those of the rectangle.

The Tietze graph can also be studied on the Möbius strip, the second strip in Figure 14 shows the graph with colouring, and the third strip contains only the graph. If we cut out the third strip and glue it together in the usual way and then we cut out regions so that only vertices and edges remain, then the Tietze graph is obtained on the surface of the Möbius strip. Count the vertices and edges of the graph and determine the degree of vertices (the number of edges that connect to each of them). Try to draw the graph on a sheet of paper. We can see that it is not a planar graph as it cannot be drawn so that its edges only intersect at their endpoints. On a Möbius strip no edges cross each other that only intersect at their endpoints.

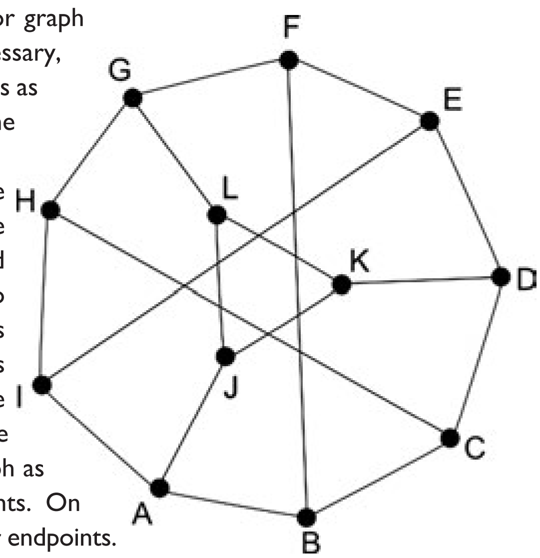


Figure 3: A potential appearance of the Tietze graph in a plane

Isomorphism and Colouring of the Tietze Graph

What does it mean in the description of Figure 3 there is 'a potential appearance of a graph'? Can it be drawn in a different way? If two graphs seem to be different, it is possible that they represent the same thing. In such a case we say that the two graphs are isomorphic. Two graphs G and H are isomorphic if there is a bijective map f from the vertices of G to the vertices of H that preserves the "edge structure" in the sense that there is an edge from vertex u to vertex v in G if, and only if, there is an edge from $f(u)$ to $f(v)$ in H. It is easy to construct isomorphic graphs, for example in a GeoGebra computer application (find GeoGebra here: www.geogebra.org). Draw the third Figure in GeoGebra. While giving instructions use only point and edge but no polygon. This is important because the graph can be formed/deformed arbitrarily in this way that is, isomorphic graphs can be drawn easily. Create isomorphic graphs with some having an axis of symmetry, rotational symmetry of order three and non-symmetric between them.

Colour vertices of the graph so that no two endpoints of an edge have the same colour (the edges should be black). What is the minimum number of colours required? The number obtained in this way is called the chromatic number of the graph. Reproducing Figure 14 allows us to try colouring.
Colour edges of the graph so that no two edges joining vertices have the same colour (now vertices should be black)! What is the minimum number of colours required? (Figure 15) The number obtained in this way is called the chromatic index of the graph. Figure 15 will help you.

Matrices Featuring the Tietze Graph

How can we store graphs using little of a computer's memory so that from the data we could unambiguously recreate and draw graphs at any time? Features of a graph are arranged in a table (it is called matrix in mathematics). Several matrices are known to store graphs. One of them is the adjacency matrix. In the first row and column of the table letters of the vertices in the graph are inputted. The rest of the table is filled in as follows. 1 is written at the place where row D and column C intersect because vertices D and C are connected by an edge, 0 is written at the place of intersection of row E and column H because vertices E and H are not connected directly by an edge. The table can be made more expressive if 0 and 1 are written using different colours and a more aesthetically pleasant, coloured table can be obtained in this way. What kind of features, symmetries can be observed in the matrix?

A more colourful and nicer figure can be obtained if a distance matrix is used. In order to fill it consider the Tietze graph again in Figure 3. At the intersection of row G and column D number 3 is written as from vertex G we can get to vertex D in 3 connecting edges. As the graph is connected we can get to each vertex along connecting edges so 0 is put only in the main diagonal of the table. Different colours are assigned again to different numbers and in this way the graph can be featured again with a new and more colourful table. Observe symmetries and rules even in this matrix! Fill or colour in the table in Figure 17 to feature the adjacency, distance matrix with numbers or colours. Hence four different expressions of a graph can be provided by a table.

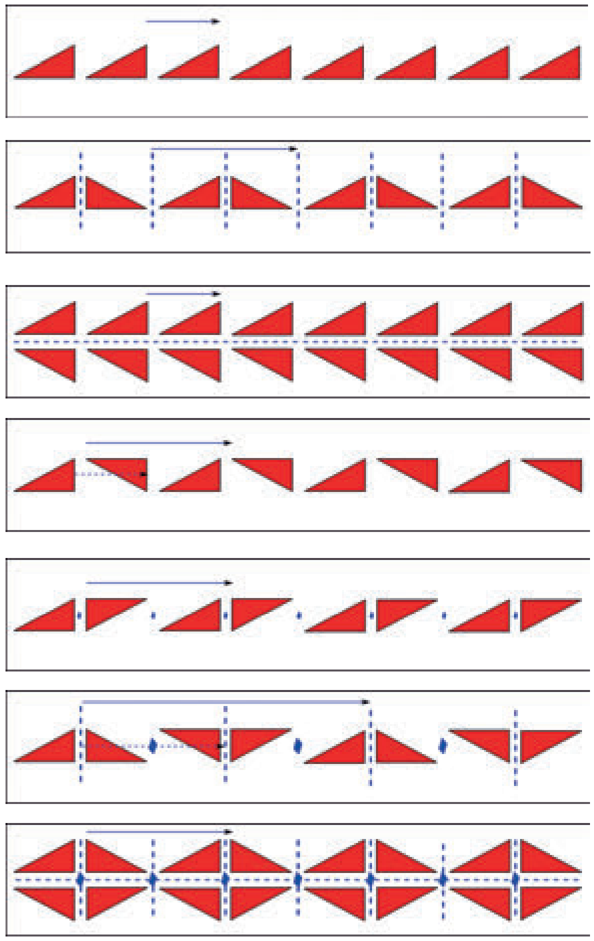
The third table is the incidence (point-edge) matrix. Here edges featured by their endpoints are written in the first row and vertices are written in the columns. Number 1 is written in the row of vertices E and F in the column of the edge EF and 0 is written everywhere else. Therefore this matrix shows which two vertices belong to a given edge. This matrix can also be coloured. By making two copies of Figure 18 a version of the incidence matrix filled with numbers and coloured in can be obtained.

Figure 4 A map colouring of the Möbius strip, graph and matrices belonging to it illustrated on a knitted jumper

Frieze Symmetries and Line Patterns

Grouping of planar symmetries can be found in Slavik Jablan and Ljiljana Radovic's chapter in this book.
The discrete group of symmetry comprising one translation is called the frieze group, or simply line patterns, patterns on a strip. Infinite line patterns, that is frieze groups generated by repeating a motive (infinitely many times), can be created in exactly seven different ways.

Grouping of the seven frieze groups can also be found in Slavik Jablan and Ljiljana Radovic's chapter in this book. Therefore here we only illustrate a list revealing all the transformations leaving the pattern of an infinite strip invariant up to each symmetry pattern, as well as demonstrating these transformations.



11 – Translation

m1 – Translation followed by a reflexion through an axis perpendicular to the direction of the translation where the distance between the axes of symmetry is half of that of the translation vector

1m – Translation followed by a reflexion through an axis parallel to the translation vector and glide reflections

1g – Translation and translation-reflection, the broken arrow denotes the glide reflection and its length is half of that of the translation

12 – Translation and rotation with an angle of 180°, the distance between centres of rotation with an angle of 180° is half of that of the translation vector

mg – Translation, rotation with an angle of 180°, reflection through an axis of symmetry perpendicular to the direction of the translation and glide reflection

mm – Translation, rotation with an angle of 180°, reflection through an axis perpendicular to the direction of the translation, reflection to an axis parallel to the translation and glide reflection, where the centre of the rotation with an angle of 180° is in the point of intersection of the horizontal and vertical axes of symmetry

Cutting out paper strips shown in Figures 19, 20 and 21, bookmarks can be made with frieze symmetries. Certainly you can also design your own bookmark according to frieze symmetries. Freehand drawings can be made or several well-known computer programmes can be used to design frieze symmetries easily. One of these programmes is for example the Kali, which can be found on the website http://geometrygames.org/.

Möbius Strip and Frieze Symmetries

By printing patterns of Figures 15, 16 and 17 on a transparent sheet and forming a cylindrical surface or Möbius strip from the paper strips we can see whether the pattern continues in the same way in both cases (i.e. thepattern is ”made infinite”). Figure 5 shows an example where the pattern does not continue according to the rule, on the Möbius strip, however, in Figure 6 it does. Which are those frieze patterns that can be put on a Möbius strip and preserve the adequate rule according to the frieze symmetry on the entire strip and which are those thatdo not? Print each of the seven patterns on a transparent paper (transparent paper is required so that the pattern could appear equally on both sides demonstrating that the thickness of the surface is negligible), create Möbius strips from them and find those transformations that must comprise frieze groups so that a given pattern could be placed orderly on the Möbius strip.

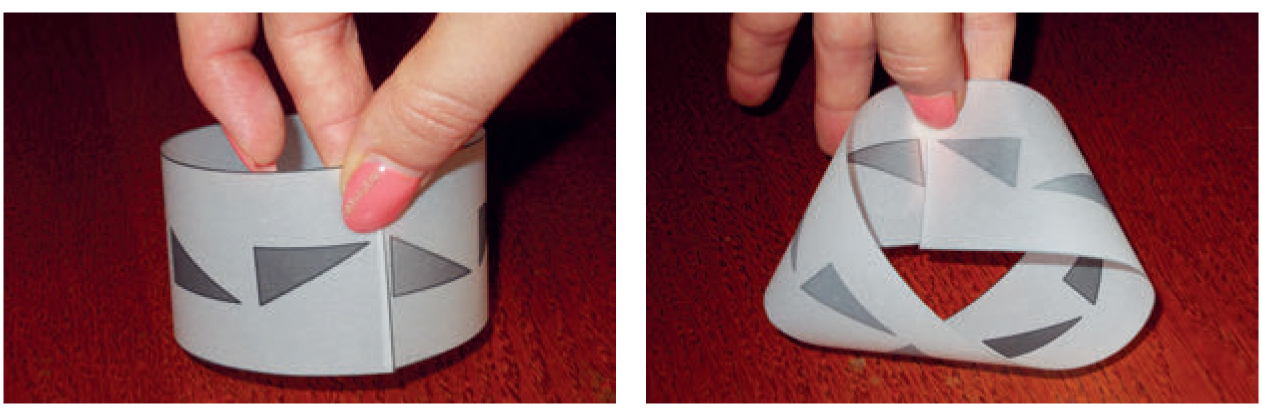


Figure 5 1g: Frieze symmetry on a cylindrical surface and on a Möbius strip

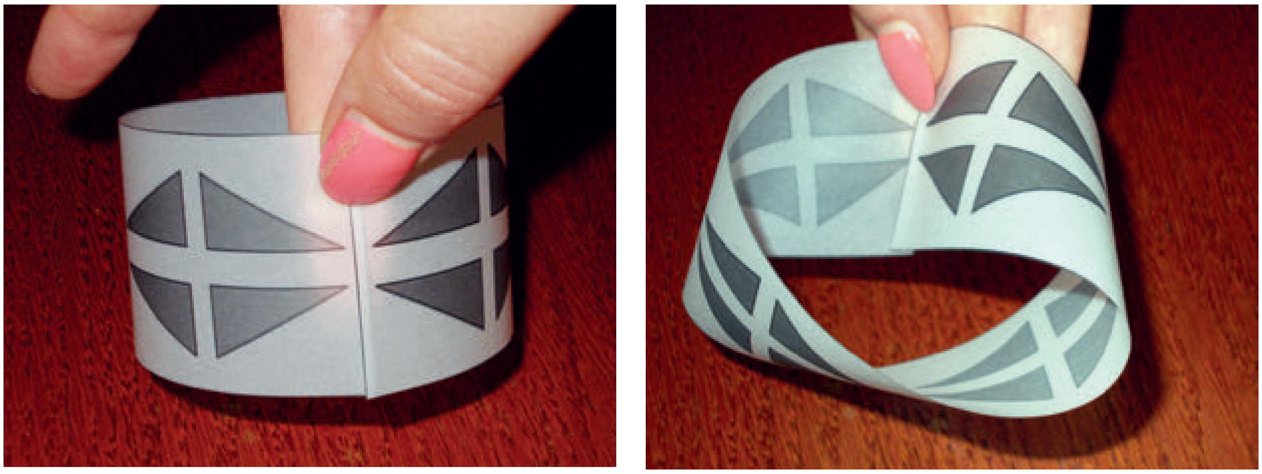


Figure 6 mm: Frieze symmetry on a cylindrical surface and on a Möbius strip

References:
http://mathworld.wolfram.com/TietzesGraph.html
http://matek.fazekas.hu/portal/tanitasianyagok/Pogats_Ferenc/sor/sorfriz.html

Illustrations and Figures:

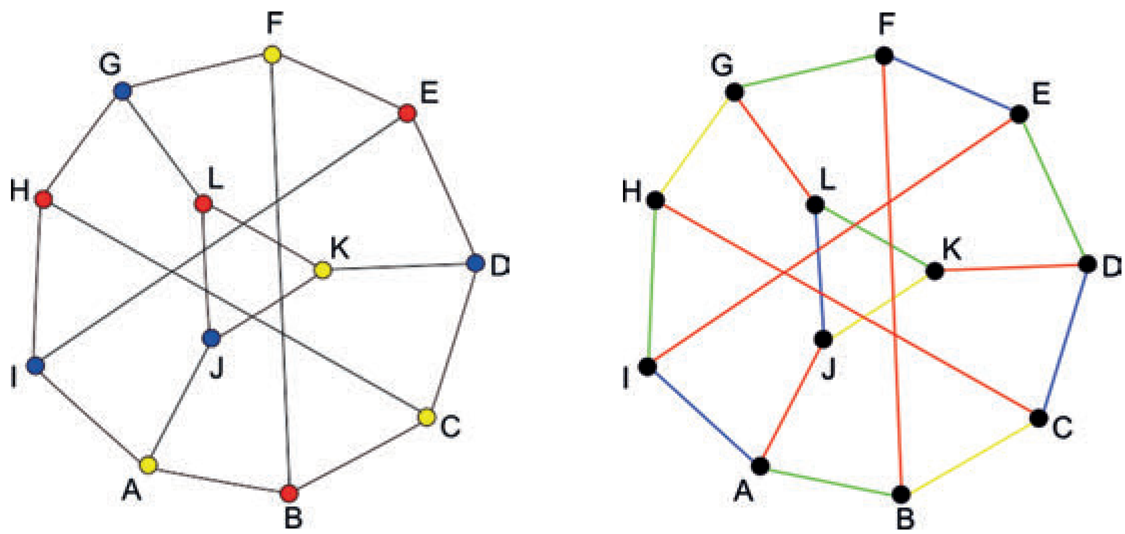


Figure 7 Illustration of the chromatic number and the chromatic index of the graph

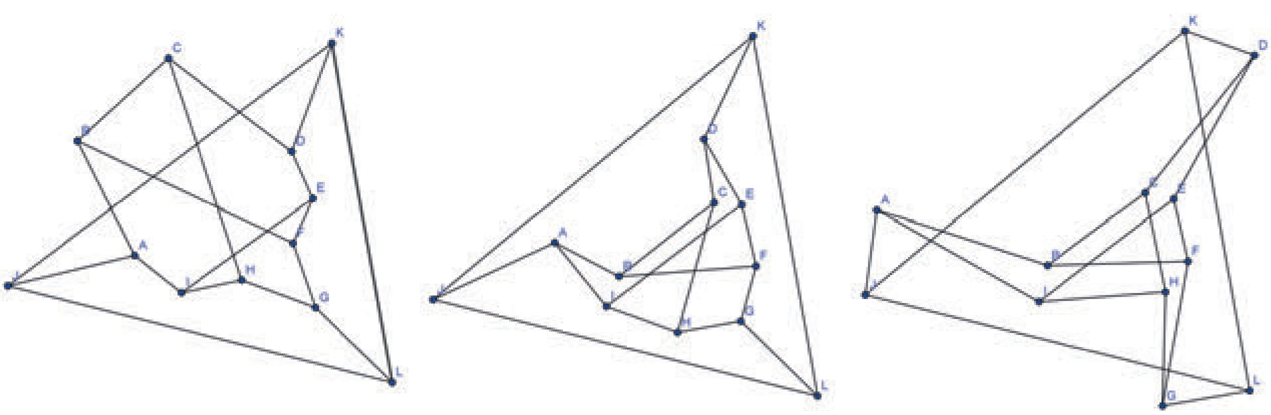


Figure 8 Some forms of appearance of the Tietze graph constructed in GeoGebra

	A	B	C	D	E	F	G	H	I	J	K	L
A	0	1	0	0	0	0	0	0	1	1	0	0
B	1	0	1	0	0	1	0	0	0	0	0	0
C	0	1	0	1	0	0	0	1	0	0	0	0
D	0	0	1	0	1	0	0	0	0	0	1	0
E	0	0	0	1	0	1	0	0	1	0	0	0
F	0	1	0	0	1	0	1	0	0	0	0	0
G	0	0	0	0	0	1	0	1	0	0	0	1
H	0	0	1	0	0	0	1	0	1	0	0	0
I	1	0	0	0	1	0	0	1	0	0	0	0
J	1	0	0	0	0	0	0	0	0	0	1	1
K	0	0	0	1	0	0	0	0	0	1	0	1
L	0	0	0	0	0	0	1	0	0	1	1	0

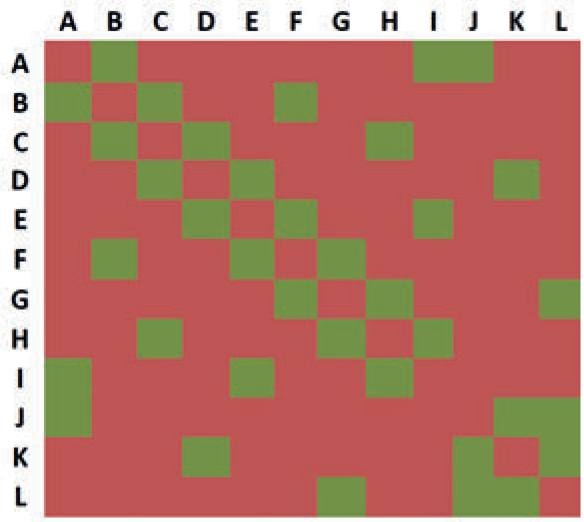


Figure 9 Adjacent matrix of the Tietze graph featured by numbers and colours

	A	B	C	D	E	F	G	H	I	J	K	L
A	0	1	2	3	2	2	3	2	1	1	2	2
B	1	0	1	2	2	1	2	2	2	2	3	3
C	2	1	0	1	2	3	2	1	2	3	2	3
D	3	2	1	0	1	2	3	2	2	2	1	2
E	2	2	2	1	0	1	2	3	1	3	2	3
F	2	1	3	2	1	0	1	2	3	3	3	2
G	3	2	2	3	2	1	0	1	2	2	2	1
H	2	2	1	2	3	2	1	0	1	3	3	2
I	1	2	2	2	1	3	2	1	0	2	3	3
J	1	2	3	2	3	3	2	3	2	0	1	1
K	2	3	2	1	2	3	2	3	3	1	0	1
L	2	3	3	2	3	2	1	2	3	1	1	0

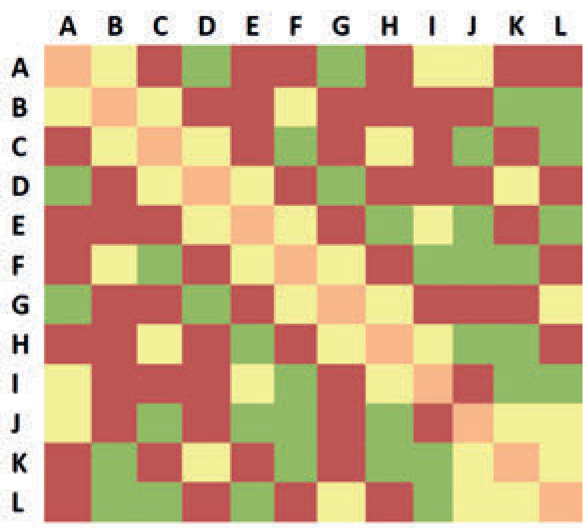


Figure 10 Distance matrix of the Tietze graph featured by numbers and colours

	A B	A I	A J	B C	B F	C D	C H	D E	D K	E F	E I	F G	G H	G L	H I	J K	J L	K L
A	1	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
B	1	0	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0	0
C	0	0	0	1	0	1	1	0	0	0	0	0	0	0	0	0	0	0
D	0	0	0	0	0	1	0	1	1	0	0	0	0	0	0	0	0	0
E	0	0	0	0	0	0	0	1	0	1	1	0	0	0	0	0	0	0
F	0	0	0	0	1	0	0	0	0	1	0	1	0	0	0	0	0	0
G	0	0	0	0	0	0	0	0	0	0	0	1	1	1	0	0	0	0
H	0	0	0	0	0	0	1	0	0	0	0	0	1	0	1	0	0	0
I	0	1	0	0	0	0	0	0	0	0	1	0	0	0	1	0	0	0
J	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	1	1	0
K	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	1	0	1
L	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	1	1

Figure 11 Incidence matrix of the Tietze graph featured by numbers

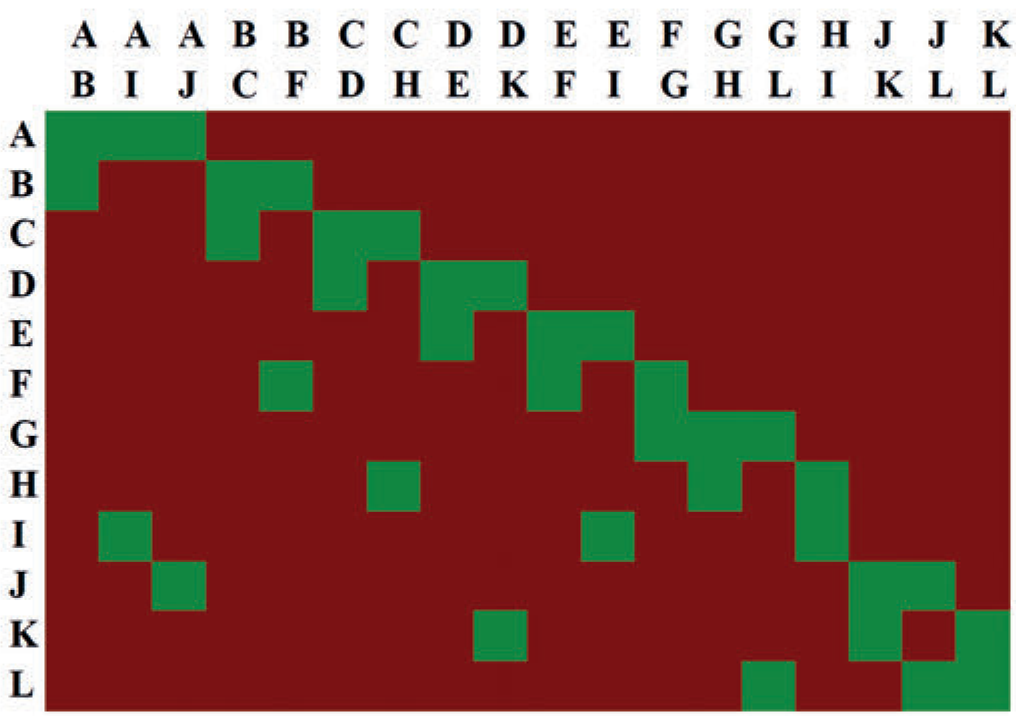


Figure 12 Incidence matrix of the Tietze graph featured by colours

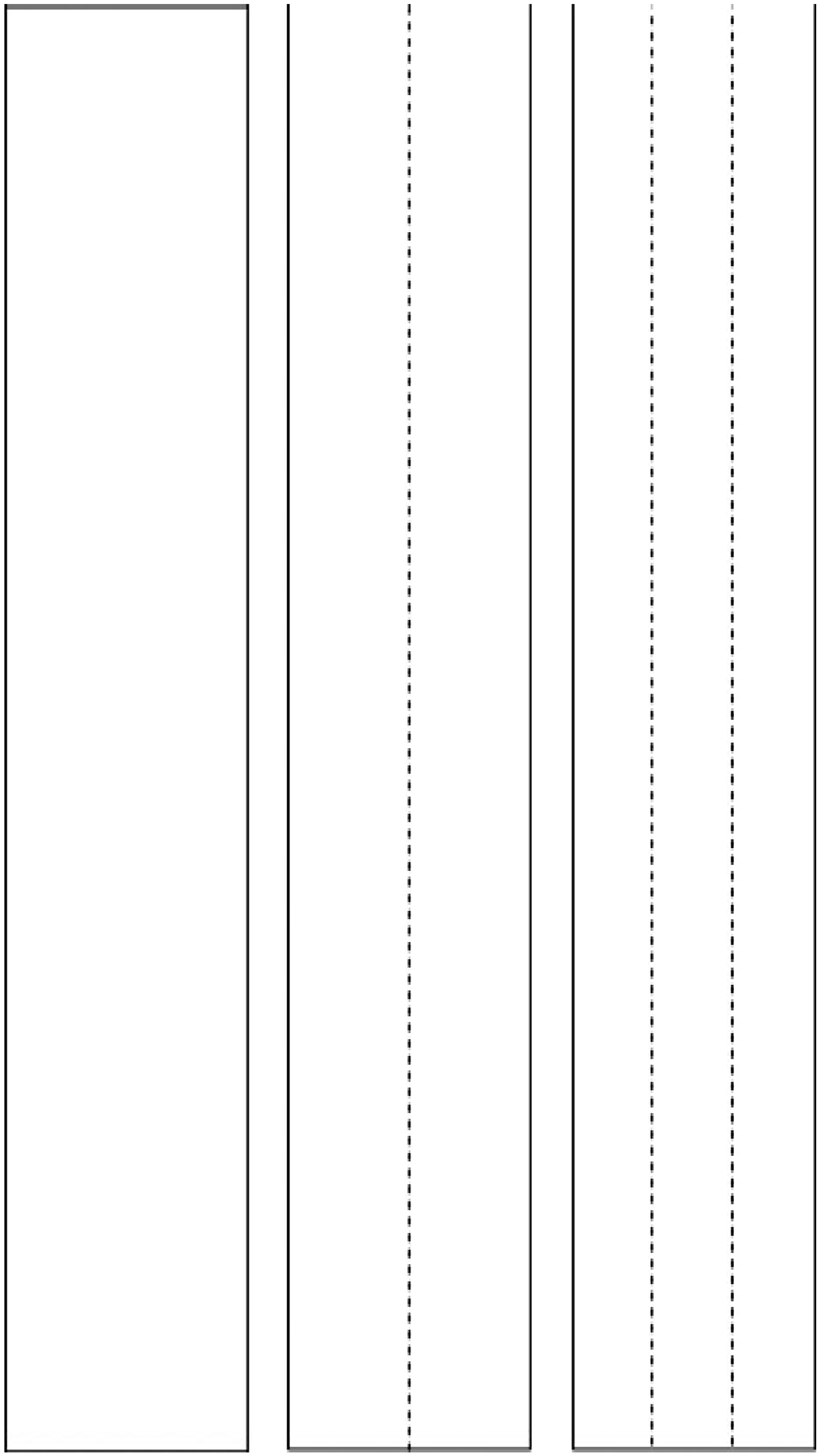


Figure 13 Möbius strip, cutting the Möbius strip along the midline and along the trisecting line

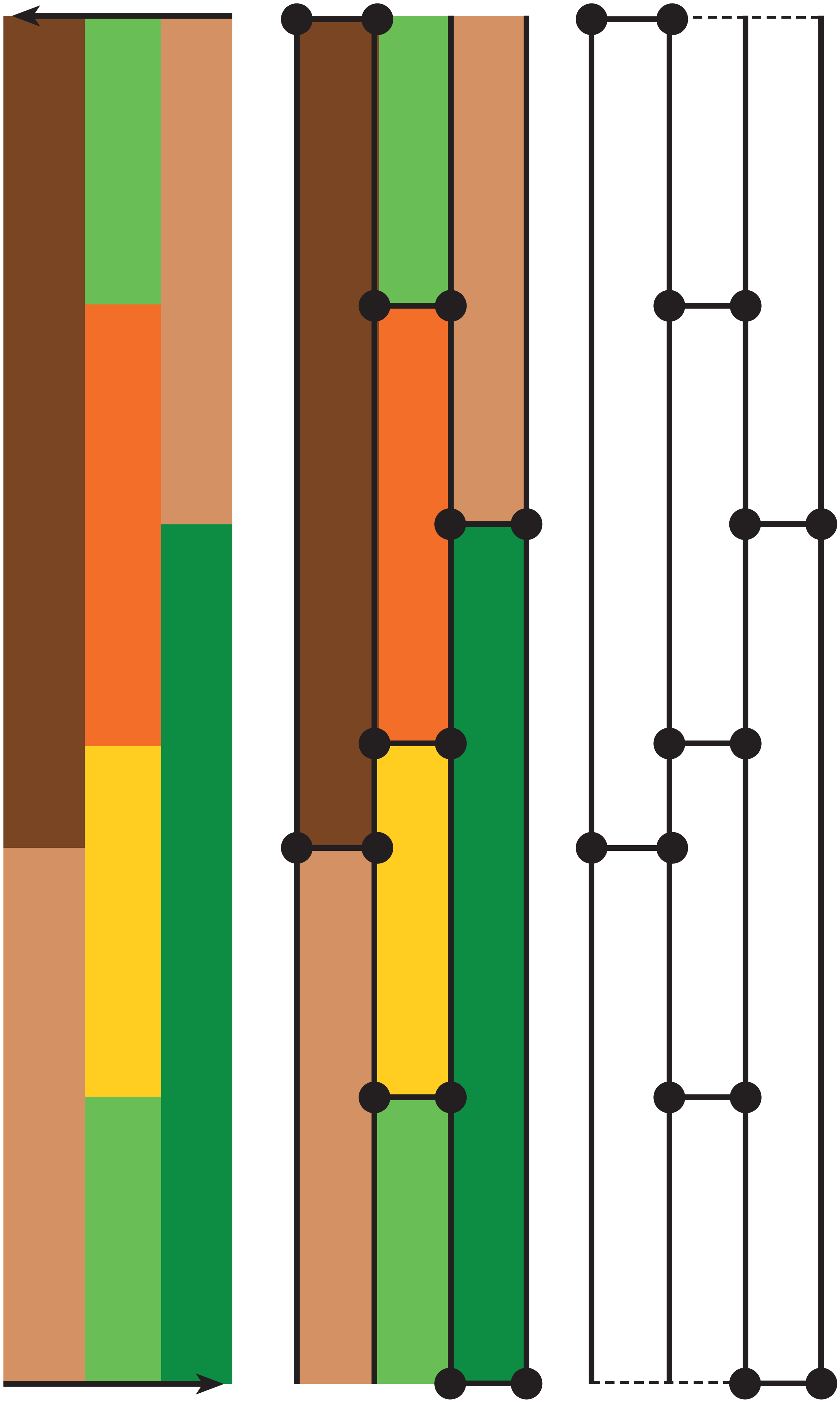


Figure 14 A “map colouring” of the Möbius strip and the Tietze graph on the Möbius strip

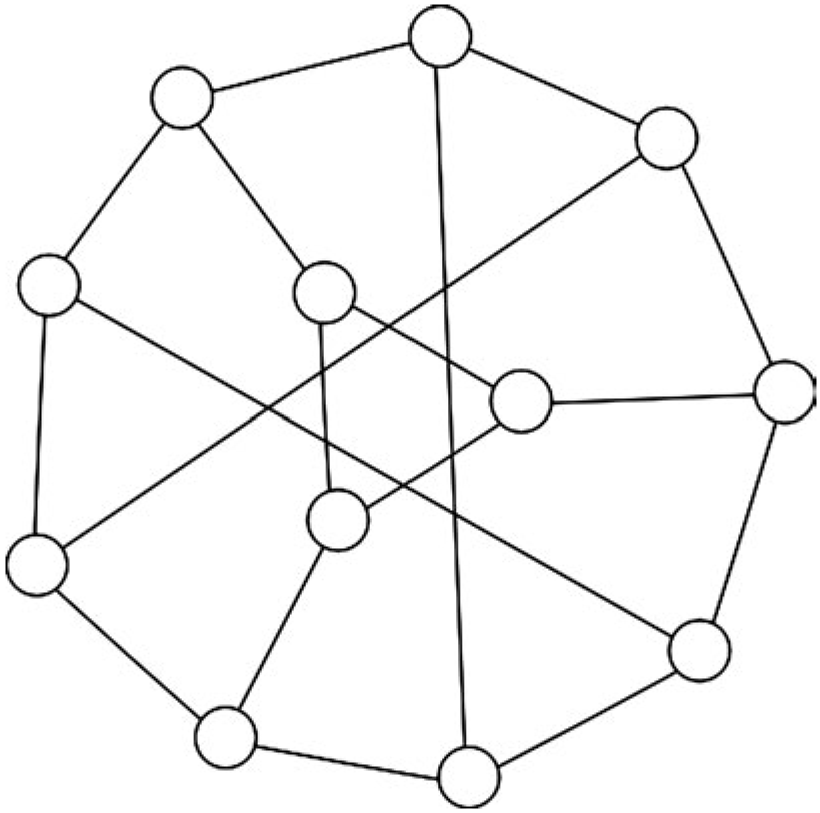


Figure 15 Illustration to determine the chromatic number of the Tietze graph

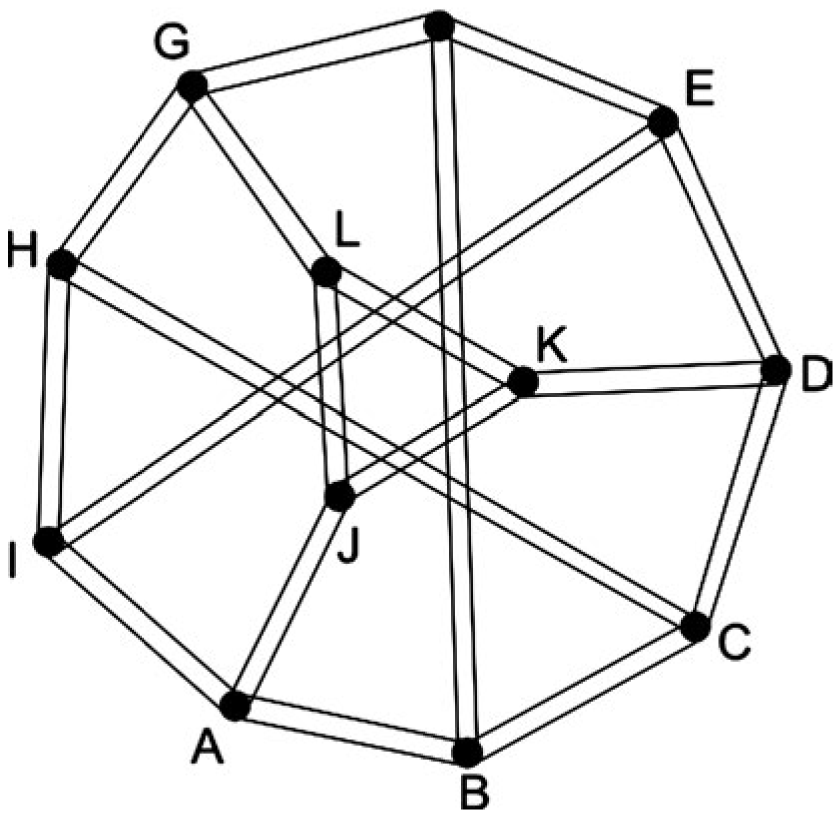


Figure 16 Illustration to determine the chromatic index of the Tietze graph

	A	B	C	D	E	F	G	H	I	J	K	L
A												
B												
C												
D												
E												
F												
G												
H												
I												
J												
K												
L												

Figure 17 Table to feature the adjacency and distance matrices of the Tietze graph with numbers and colours

	A B	A I	A J	B C	B F	C D	C H	D E	D K	E F	E I	F G	G H	G L	H I	J K	J L	K L
A																		
B																		
C																		
D																		
E																		
F																		
G																		
H																		
I																		
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Figure 18 Table to feature the incidence matrix of the Tietze graph with numbers and colours

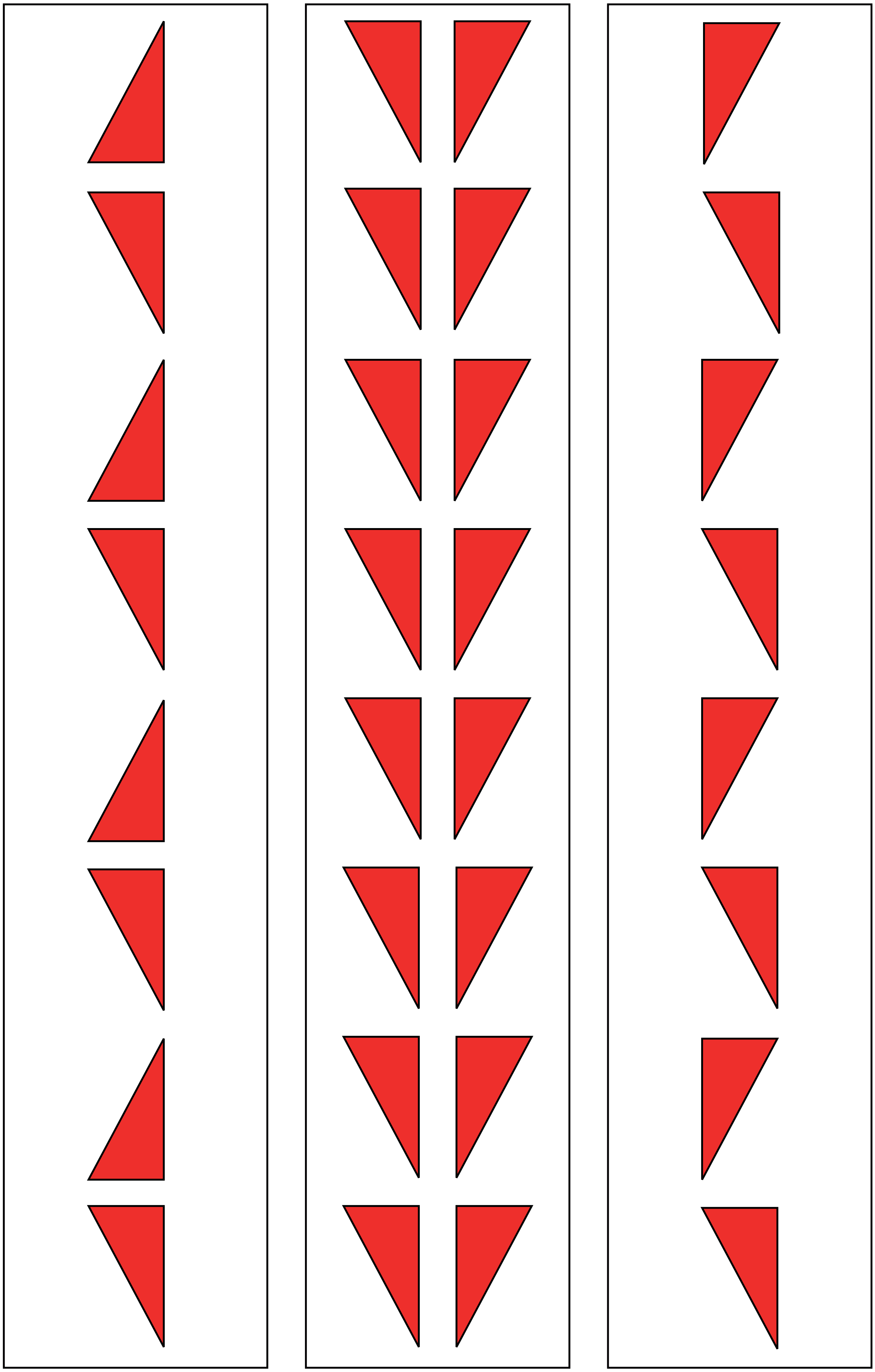


Figure 19 1g, 1m, m1: frieze groups

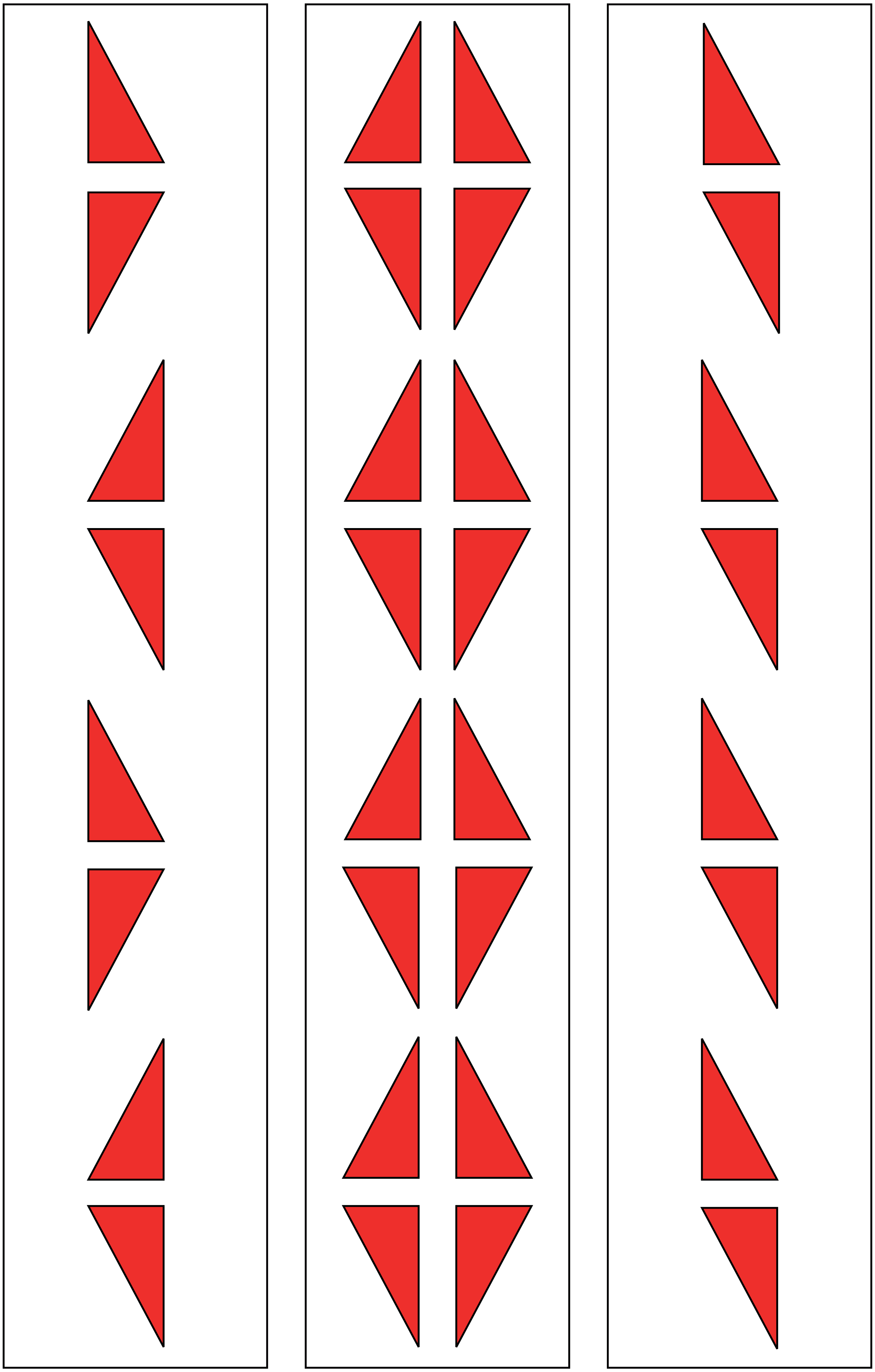


Figure 20 mg, mm: twelve frieze groups

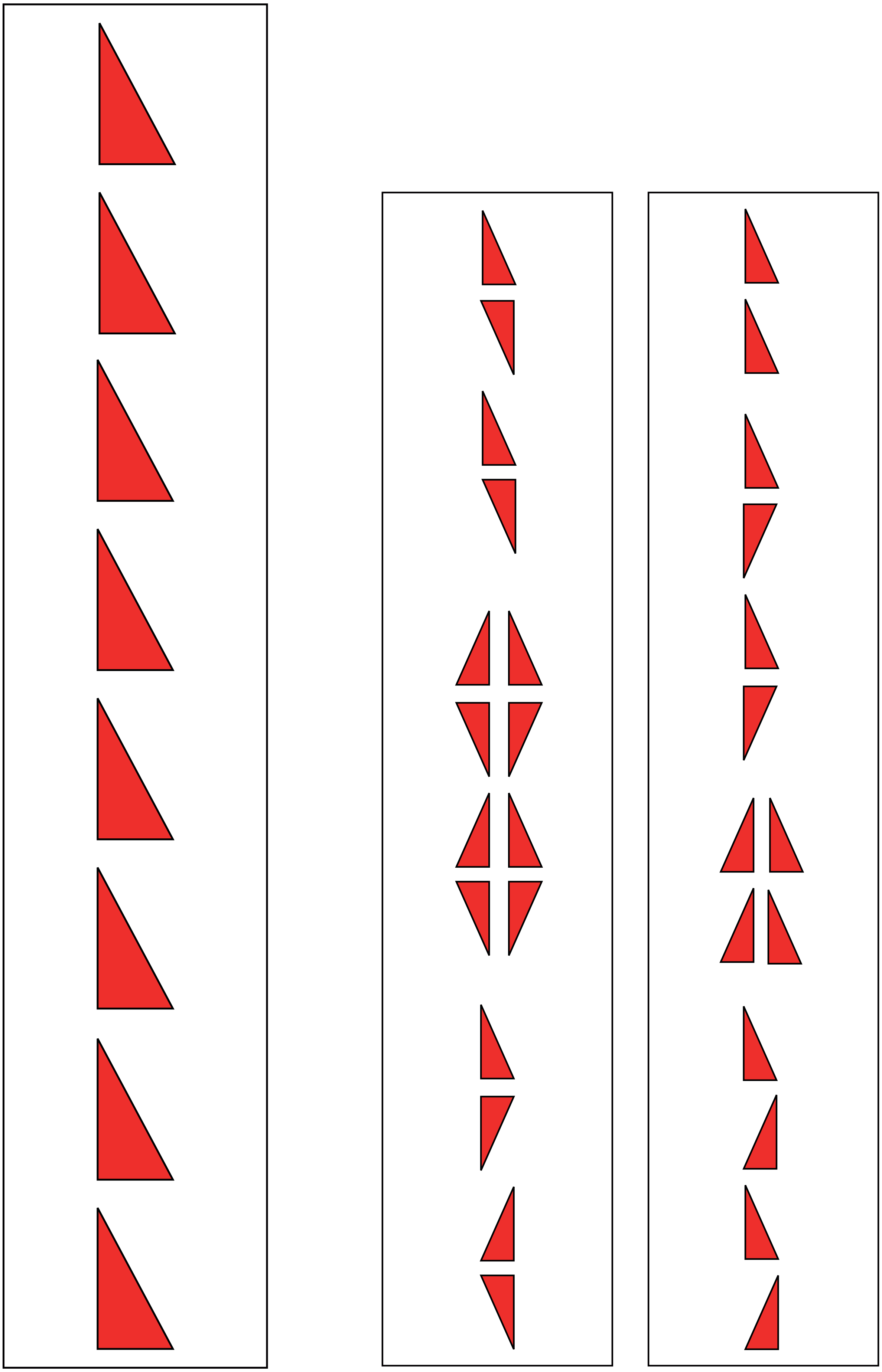
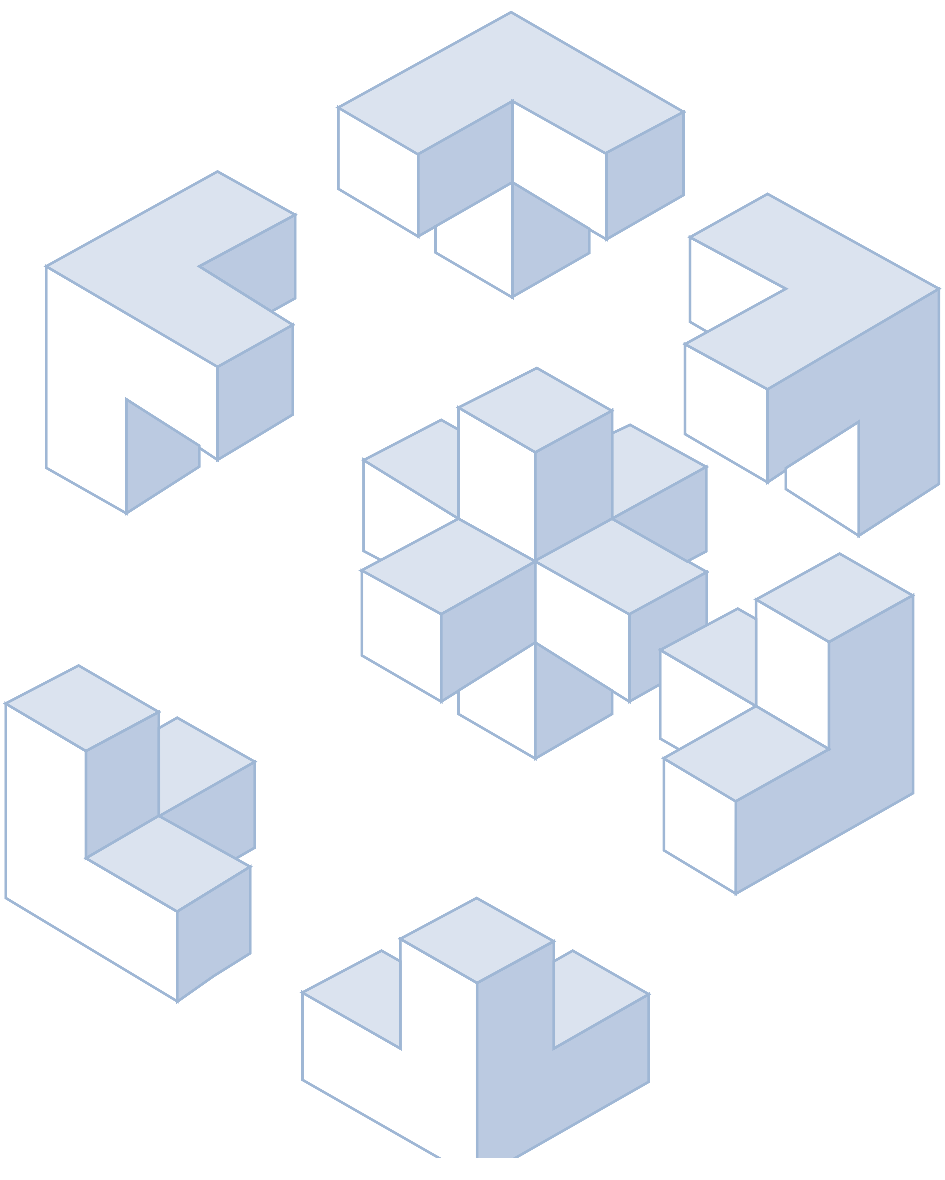
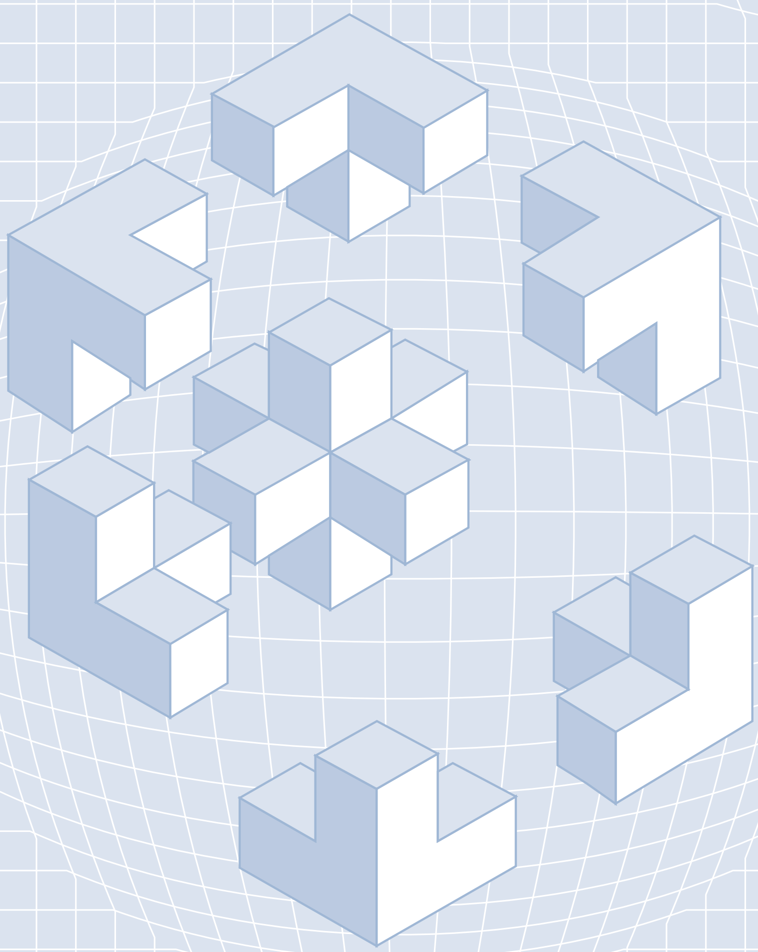


Figure 21: 11 frieze groups and 7 frieze patterns together on a bookmark



Jay Bonner

The Polygonal Technique in Islamic Geometric Design



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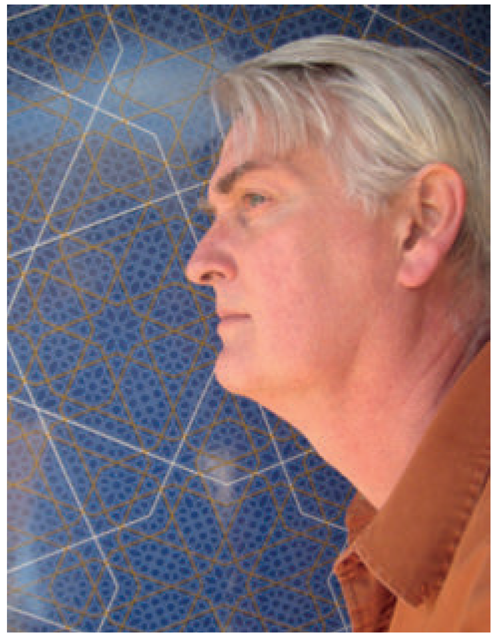


JAY BONNER is a specialist in the field of Islamic architectural ornament, and is a recognized authority on Islamic geometric design methodology. His many ornamental projects include work on the Grand Mosque in Mecca, and the expansion of the Prophet's Mosque in Medina, Saudi Arabia. He has taught seminars and lectured widely on the subject of Islamic geometric design methodology, including in Europe, the USA, Morocco and Turkey. His publications on design methodology are available for free download from the archive of the Bridges Math and Art website at http://archive.bridgesmathart.org.
Jay Bonner's book on Islamic geometric design methodology and historical development is being published by Springer Verlag, and is scheduled for release in early 2015. Examples of his work can be seen at his website: www.bonner-design.com.

Introduction to the Activities

Islamic geometric design is a particularly sophisticated visual expression of mathematical principles. Muslim artists were innovators of new forms of filling the two-dimensional plane with ever more complex patterns that included not just isometric and orthogonal symmetries, but 5-fold and 7-fold symmetrical systems as well. By the 14th and 15th centuries geometric designs were being produced that fulfill the modern mathematical criteria of self-similarity.

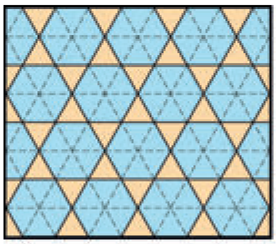
The principle historical method of creating Islamic geometric patterns is the *polygonal technique*. This uses key points upon a polygonal tessellation to construct a design. After the pattern has been completed the polygonal tessellation is discarded, leaving only the geometric design. Because the tessellation does not form part of the completed geometric pattern, it is referred to as an underlying tessellation. The construction of most geometric designs involves the placement of crossing pattern lines at the midpoints of the polygonal edges within a given tessellation. Depending upon the angle of the crossing pattern lines, different aesthetic characteristics are achieved. The different conventions for placing the crossing pattern lines led to the development of the four pattern families that are ubiquitous to this tradition.

There are two varieties of pattern created from the polygonal technique: systematic and non-systematic. The underlying tessellations of non-systematic patterns are comprised of regular and irregular polygons that only work together in one unique arrangement. They cannot be reassembled into other tessellations. By contrast, the underlying tessellations of systematic patterns are comprised of a limited set of polygons that can be assembled into an unlimited number of different combinations. There are five distinct systems that were used historically: (1) the system of regular polygons; (2) the 4-fold system A; (3) the 4-fold system B; (4) the 5-fold system; and (5) the 7-fold system. Each of these has its own geometric characteristics and aesthetic qualities, and each is capable of tremendous design versatility and beauty. The following activities are based upon three of these systematic design methodologies.

Activity 1 – the System of Regular Polygons:

As the name suggests, the system of regular polygons is comprised of regular polygons. The illustrations below are the eight semi-regular tessellations, as well as a selection of twelve demi-regular tessellations. The semi-regular tessellations have more than one type of polygon and have identical vertices, while the demi-regular tessellations have more than one type of polygon and two varieties of vertex. All of these will create very good geometric designs in each of the three pattern families.

As the name suggests, the system of regular polygons is comprised of regular polygons. The illustrations below are the eight semi-regular tessellations, as well as a selection of twelve demi-regular tessellations. The semi-regular tessellations have more than one type of polygon and each has a single variety of vertex configuration. These are identified by their respective numeric intervals, for example: 3.6.3.6; 3.3.3.4.4; and 3.12.12, etc. The demi-regular tessellations have more than one type of polygon and two varieties of vertex; also identified by the numeric intervals of the two types of vertex, for example: 3.3.6.6 – 3.6.3.6; and 3.4.6.4 – 4.6.12, etc. All of these will create very good geometric designs in each of the three pattern families provided below.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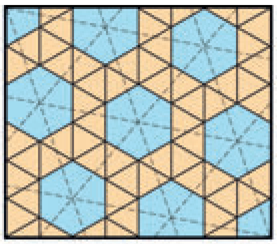
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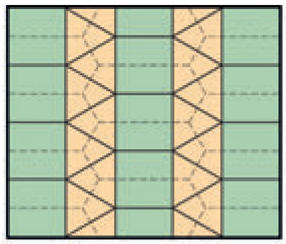
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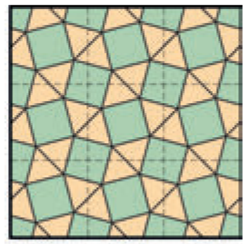
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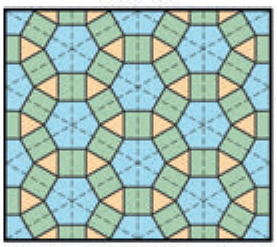
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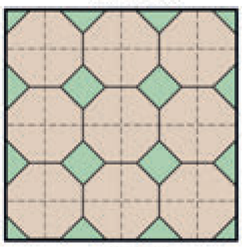
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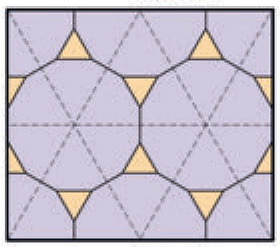
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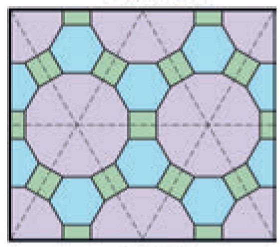
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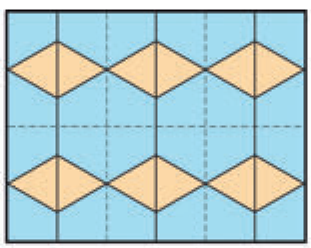
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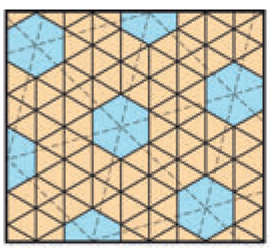
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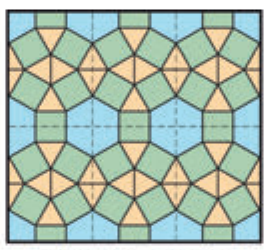
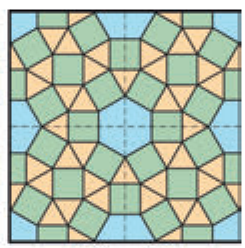

3.3.4.3.4 - 3.4.6.4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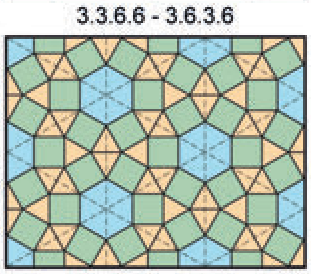
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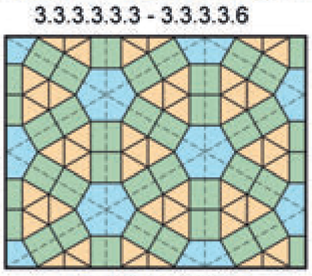
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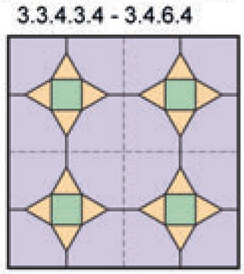
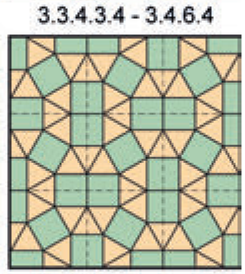
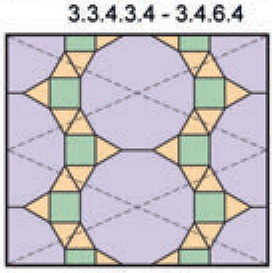
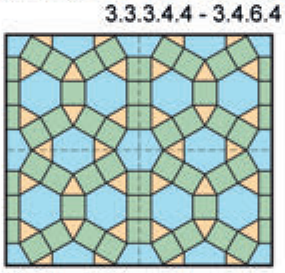
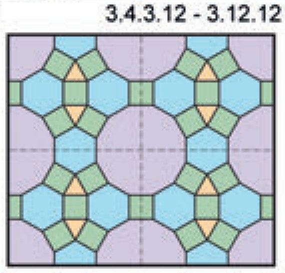
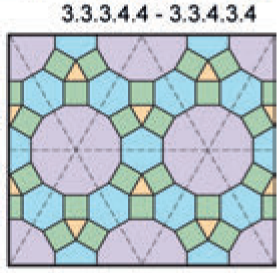

3.4.3.12 - 3.12.12


3.3.3.4.4 - 3.3.4.3.4

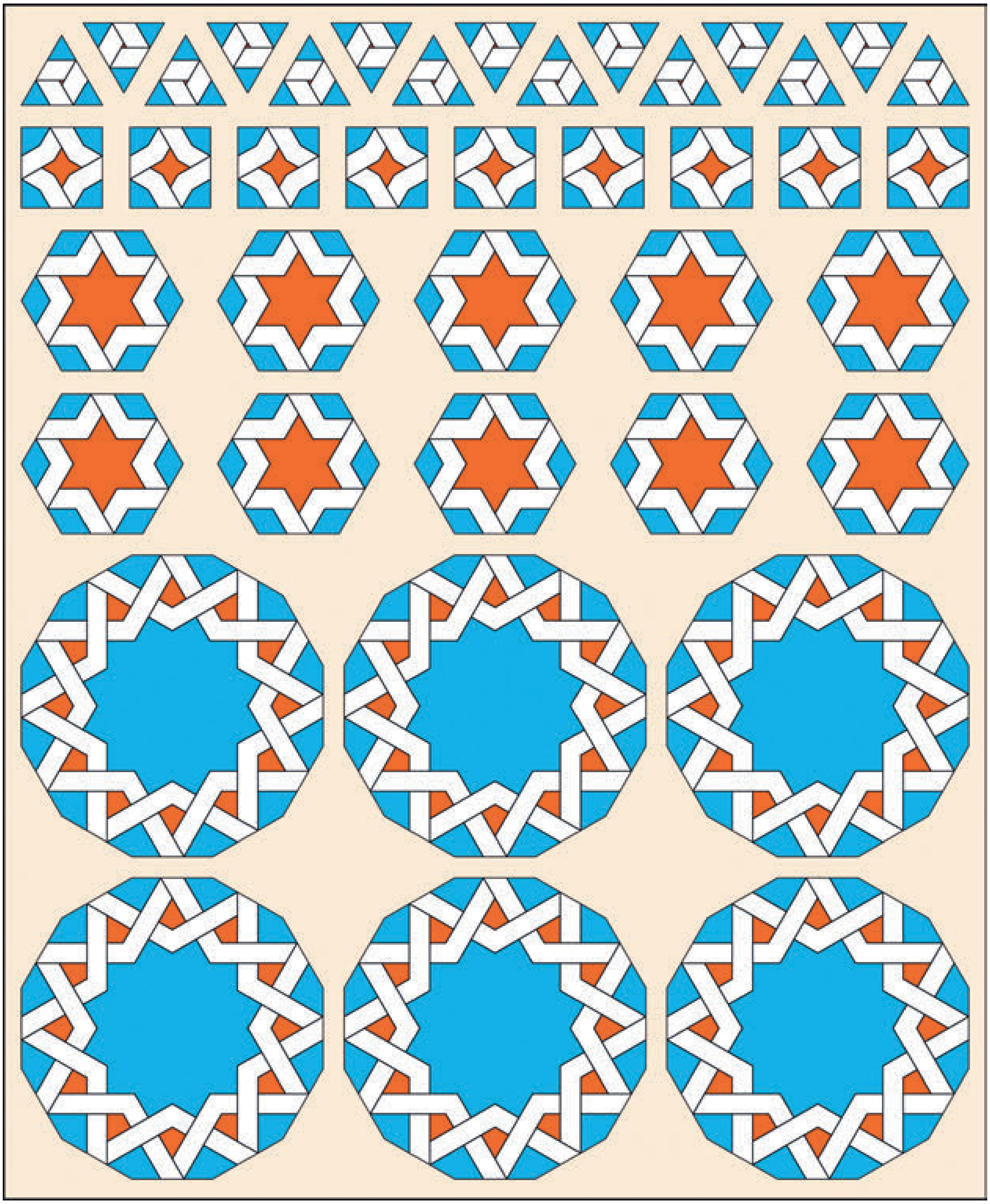

3.3.4.12 - 3.4.3.12

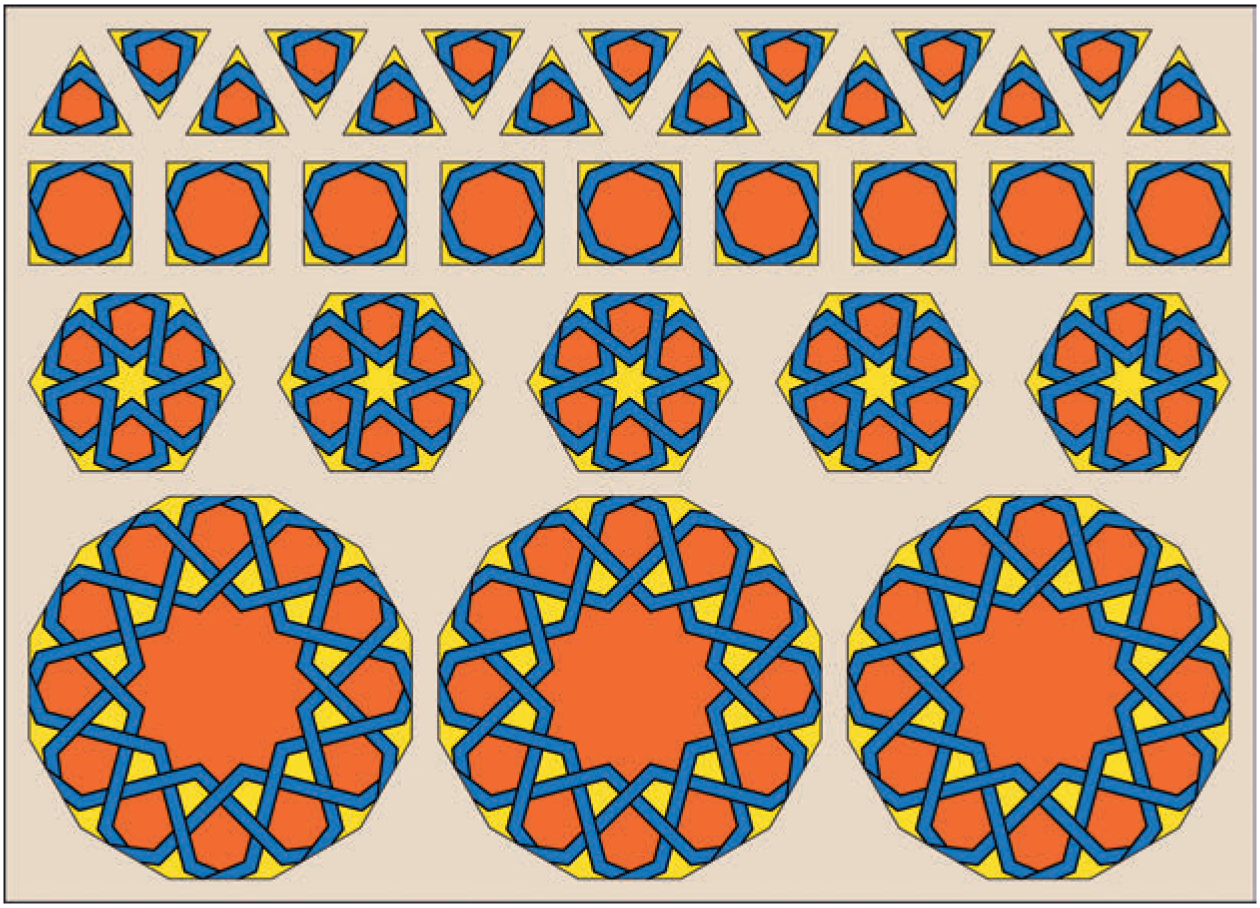

3.4.4.6 - 3.4.6.4

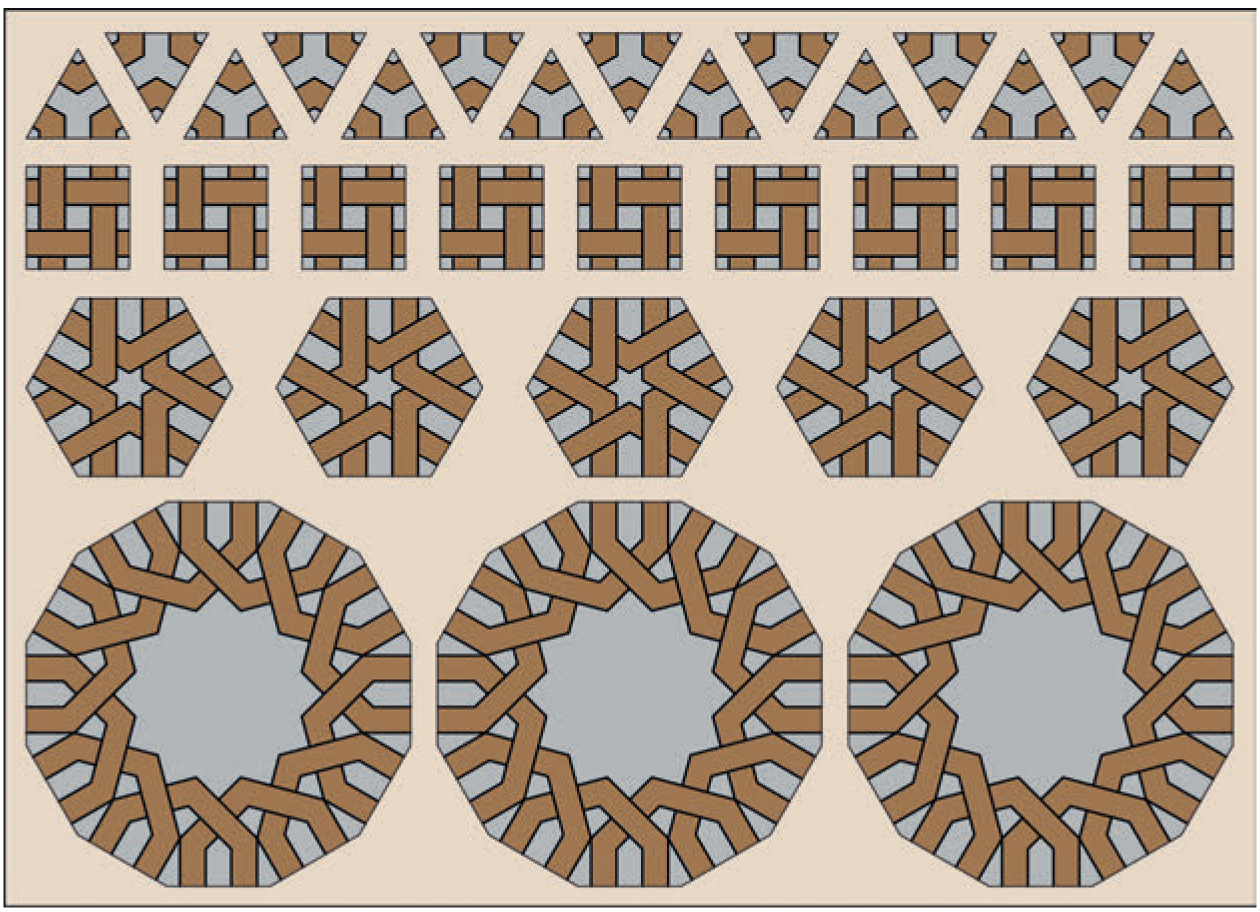

3.4.6.4 - 4.6.12


3.4.6.4 - 4.6.12

The pre-decorated polygons in this activity combine together to make attractive geometric patterns. There are three sets of polygons. The first makes *acute* patterns, the second makes *obtuse* patterns, and the third makes *2-point* patterns. Step 1: make several color photocopies of this and the following pages. Step 2: carefully cut out the polygons with a pair of scissors. Note: it helps to use heavy paper stock. Step 3: assemble the cut out polygons into geometric patterns using the semi-regular and demi-regular tessellations from the previous page. Try many different combinations. Step 4: once an appealing pattern is found, the polygons can be glued down to another piece of paper for permanence.

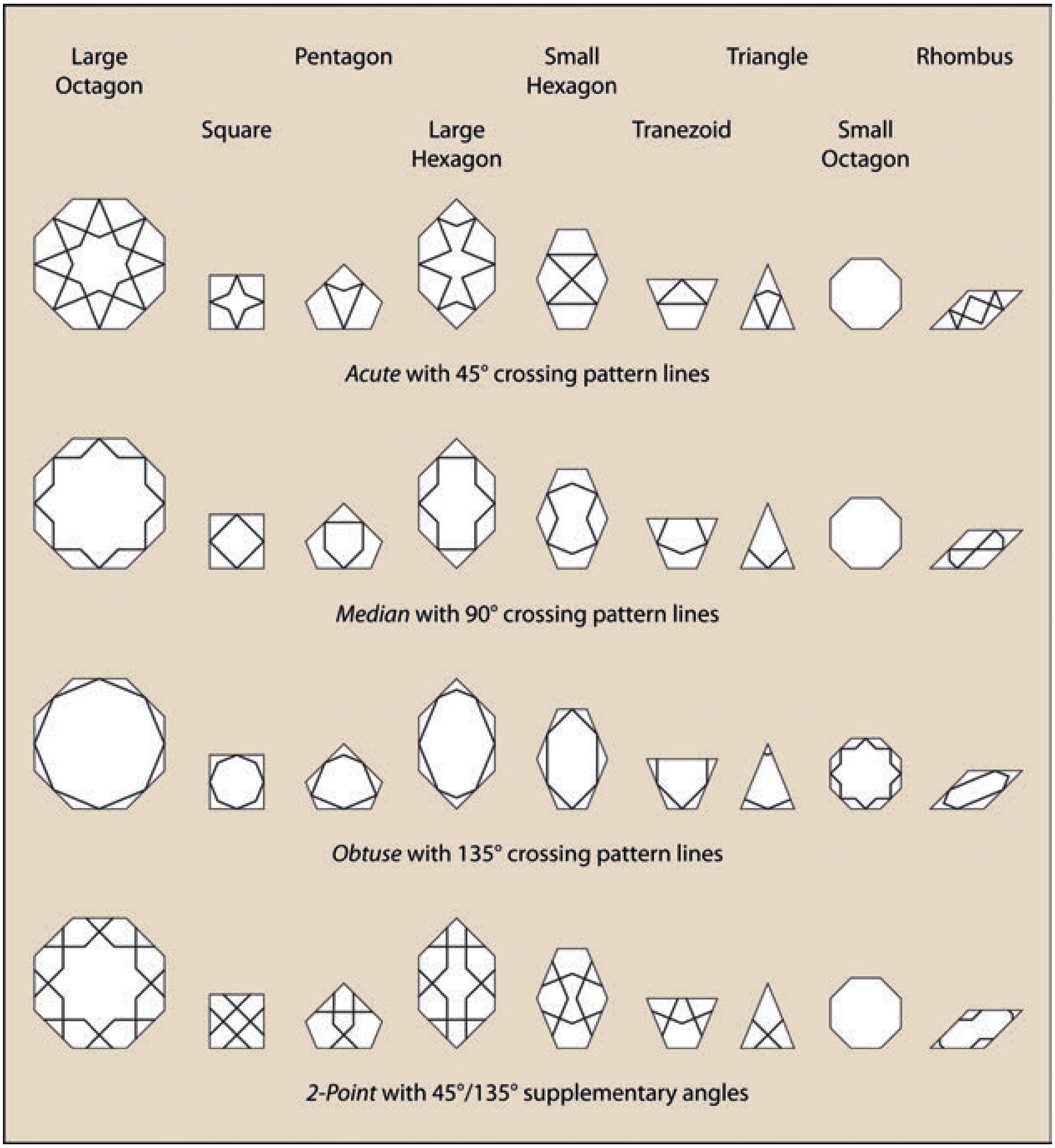




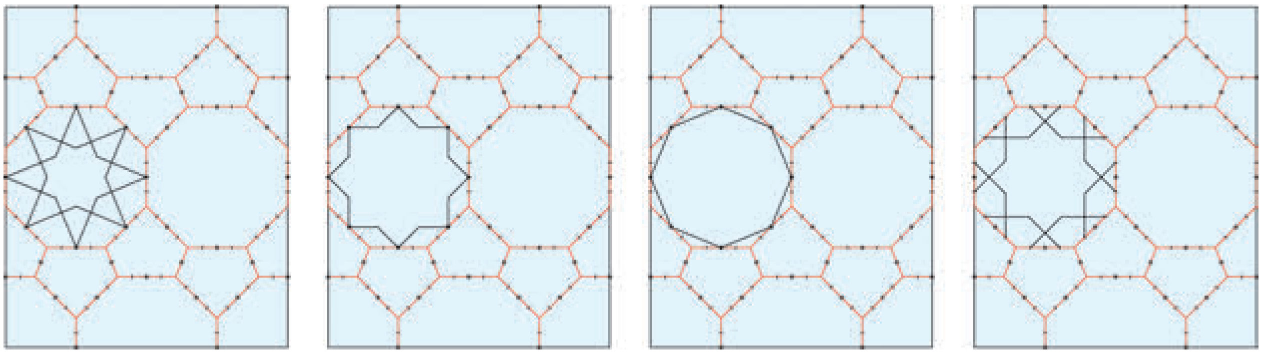


Activity 2 – the 4-Fold System A:

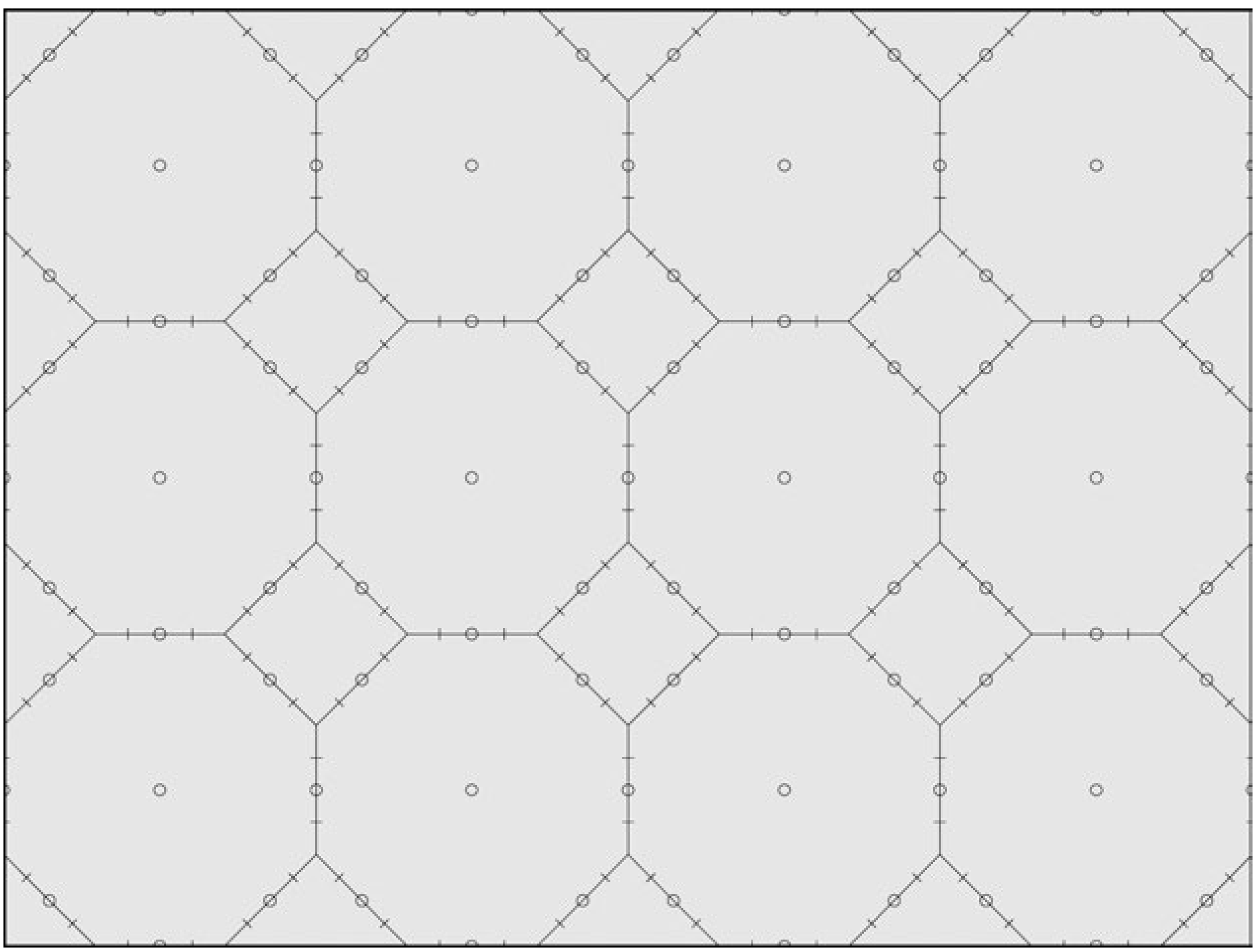
The polygonal elements that comprise this system create patterns with 4-fold symmetry. Generally these patterns have square repeat units, but sometimes rhombic, rectilinear, non-regular hexagonal and radial repeats are also encountered. The four pattern families are applied to the polygonal elements below. The primary polygon in this system is the octagon. The applied pattern lines of the *acute* family are determined by drawing a line from a midpoint of an edge of the octagon to the midpoint of the third edge in sequence. This creates a star with 45° angles. A line that connects every second midpoint determines the 90° angles of the *median* family. The angle of the *obtuse* family in this system is 135° and is produced by connecting every adjacent midpoint. The lines of the *2-point* family connect every second edge, but originate from two points along each edge. Notice how the polygonal elements have two varieties of edge length. This makes the process of tessellating more interesting.



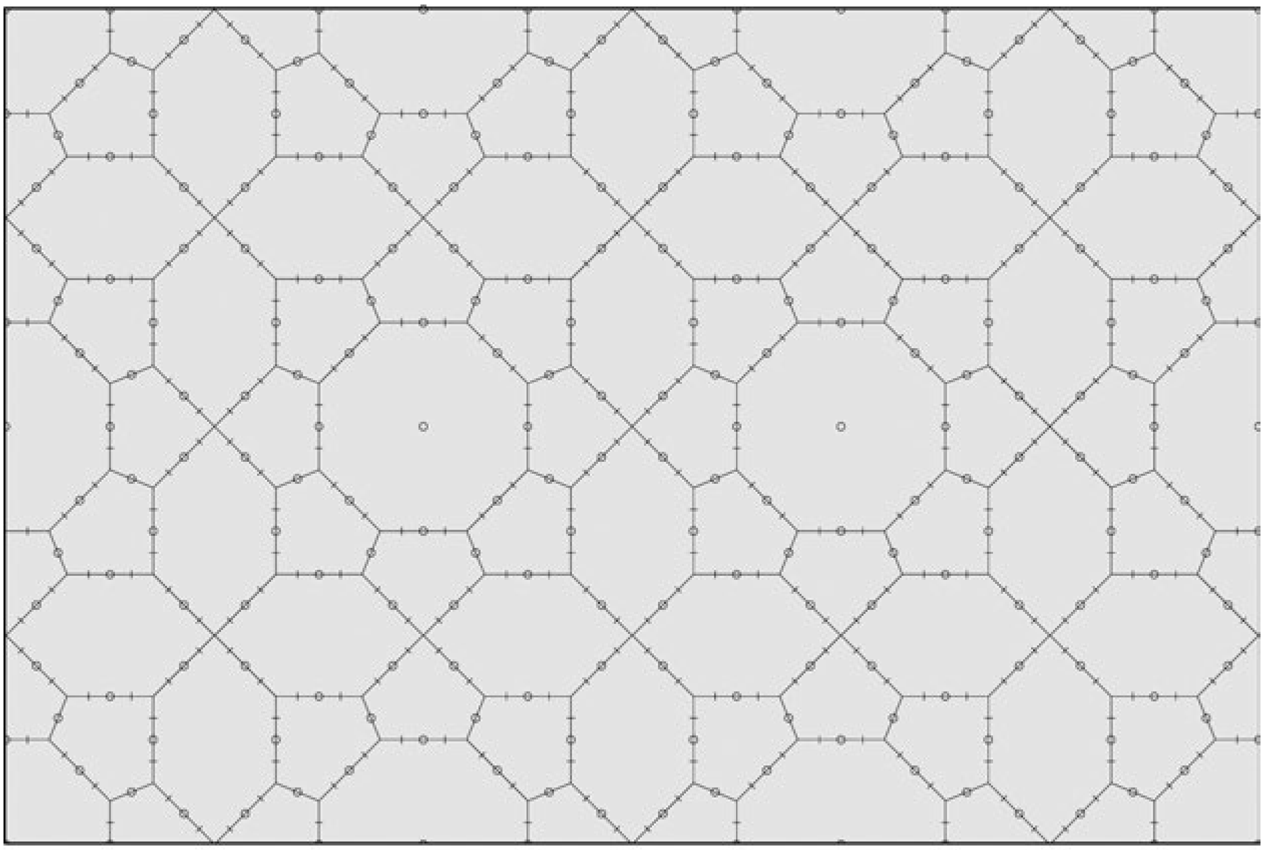
In this activity, pattern lines are drawn onto provided underlying tessellations. To assist this process, midpoint and 2-point indicators are provided on each polygonal edge. Step 1: make four photocopies of this page and the following page with the provided tessellations. Step 2: as a practice run, complete the pattern lines in the initial four tessellations below. Step 3: using the previous illustration as a guide, draw pattern lines onto each of the provided underlying tessellations in each of the four pattern families. Note: it helps to begin with drawing the pattern lines within the octagons, and to extend the lines outward from the octagons into the adjacent polygons. Step 4: color the results for added visual interest.

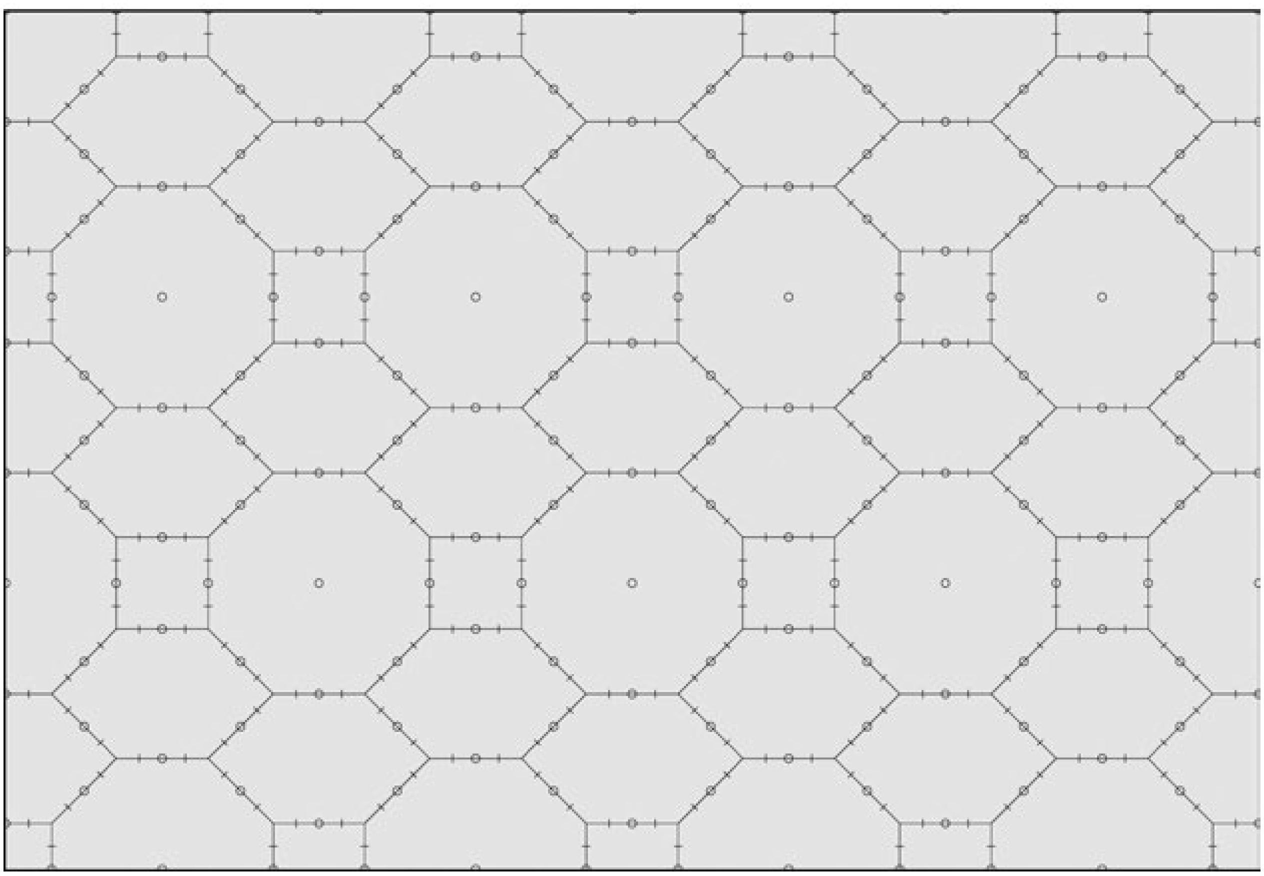


The 4.8.8 tessellation below is one of the most commonly used generative tessellations used historically to make Islamic patterns. This grid of octagons and squares makes a variety of very beautiful designs. Once you have completed making patterns in each of the four pattern families with this underlying tessellation you can experiment with your own variations to these designs. Additive arbitrary design variations were a vital part of this design tradition.



These two underlying generative tessellations make very good geometric patterns in each of the four pattern families. The upper tessellation repeats upon the orthogonal grid. The lower example is unusual in that the repeat is rhombic. As an added exercise, determine the repeat units by drawing lines connecting the centers of the octagons.



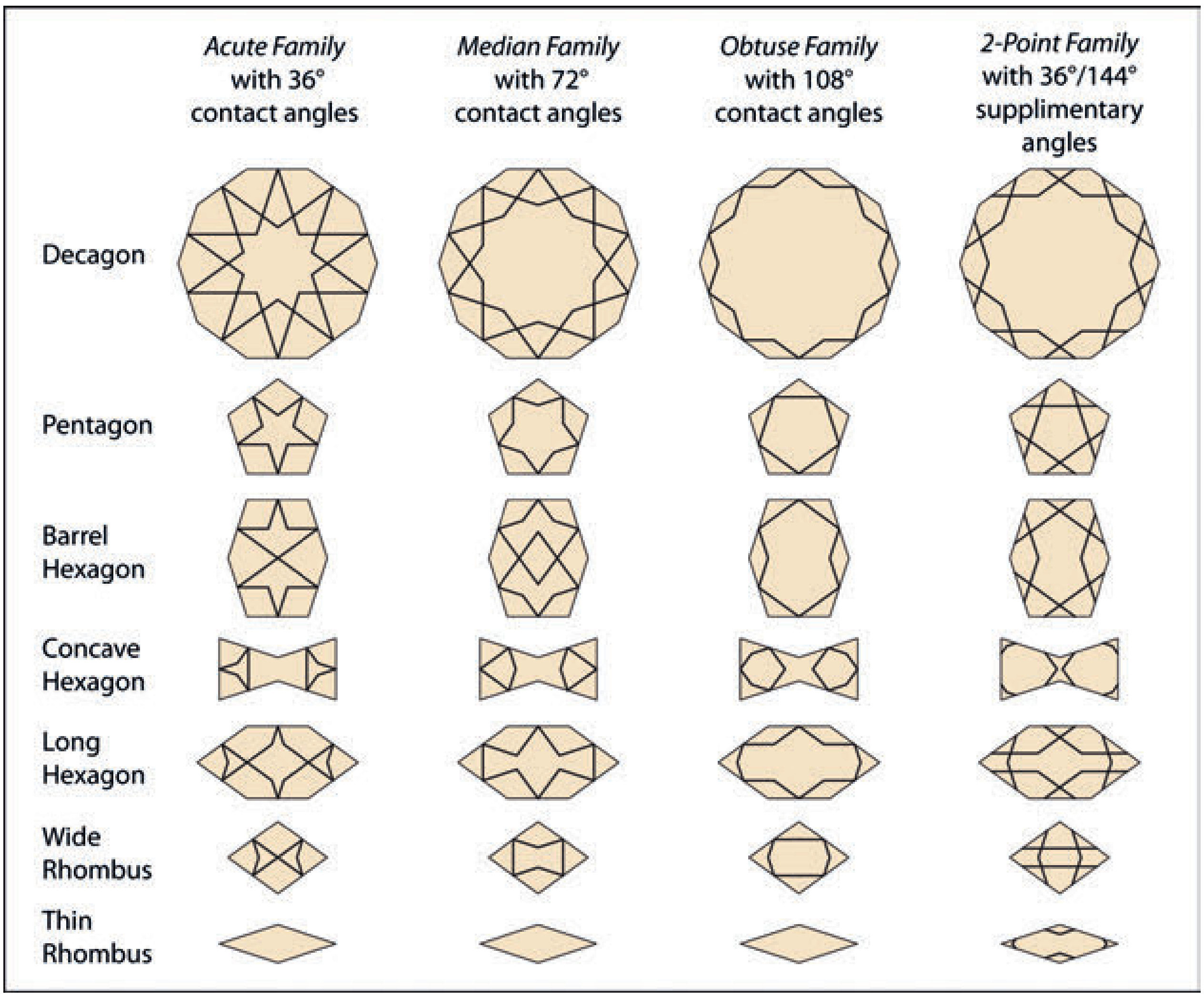


Activity 3 - The 5-Fold System:

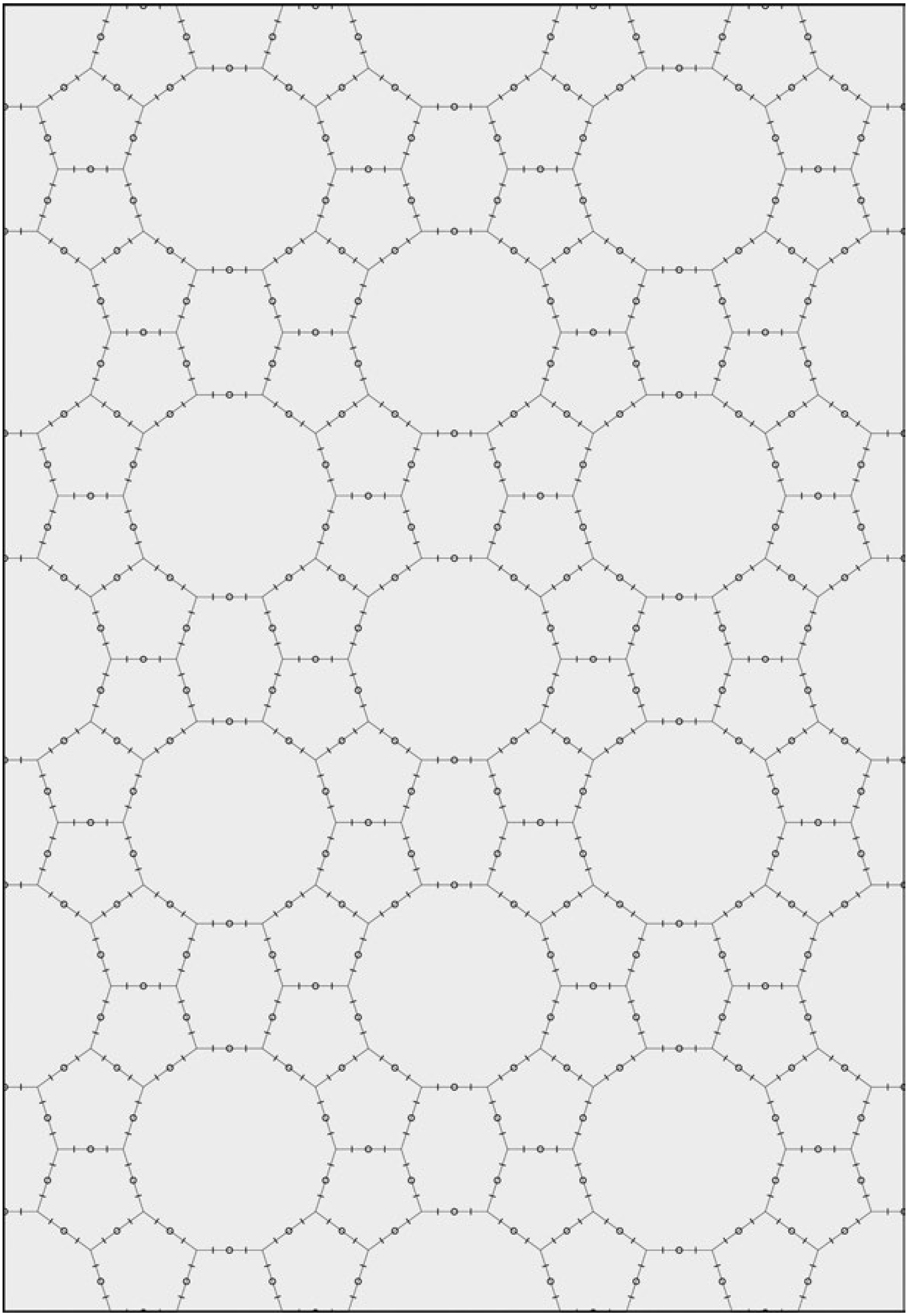
Patterns created from the *5-fold system* were among the most popular in the Islamic world. The primary polygonal elements of the *5-fold system* are the regular decagon and pentagon. The pre-decorated elements below are the most regularly used polygons, but others also exist. Drawing a line from the midpoint of an edge of the decagon and connecting it to the fourth midpoint in sequence determines the 36° crossing pattern lines of the *acute* family. The angles of the 72° *median* family skips three edges, and those of the *obtuse* skip two edges. The *2-point* patterns also skip two edges, but employ two points on each polygonal edge. Each of these pattern types produces very beautiful geometric patterns.

Activity 3A - Step 1: make four photocopies of the tessellation from the following page. Step 2: using the illustration below as a guide, draw the pattern lines onto the photocopied underlying tessellations in each of the four pattern families. When finished, color the designs for visual interest.

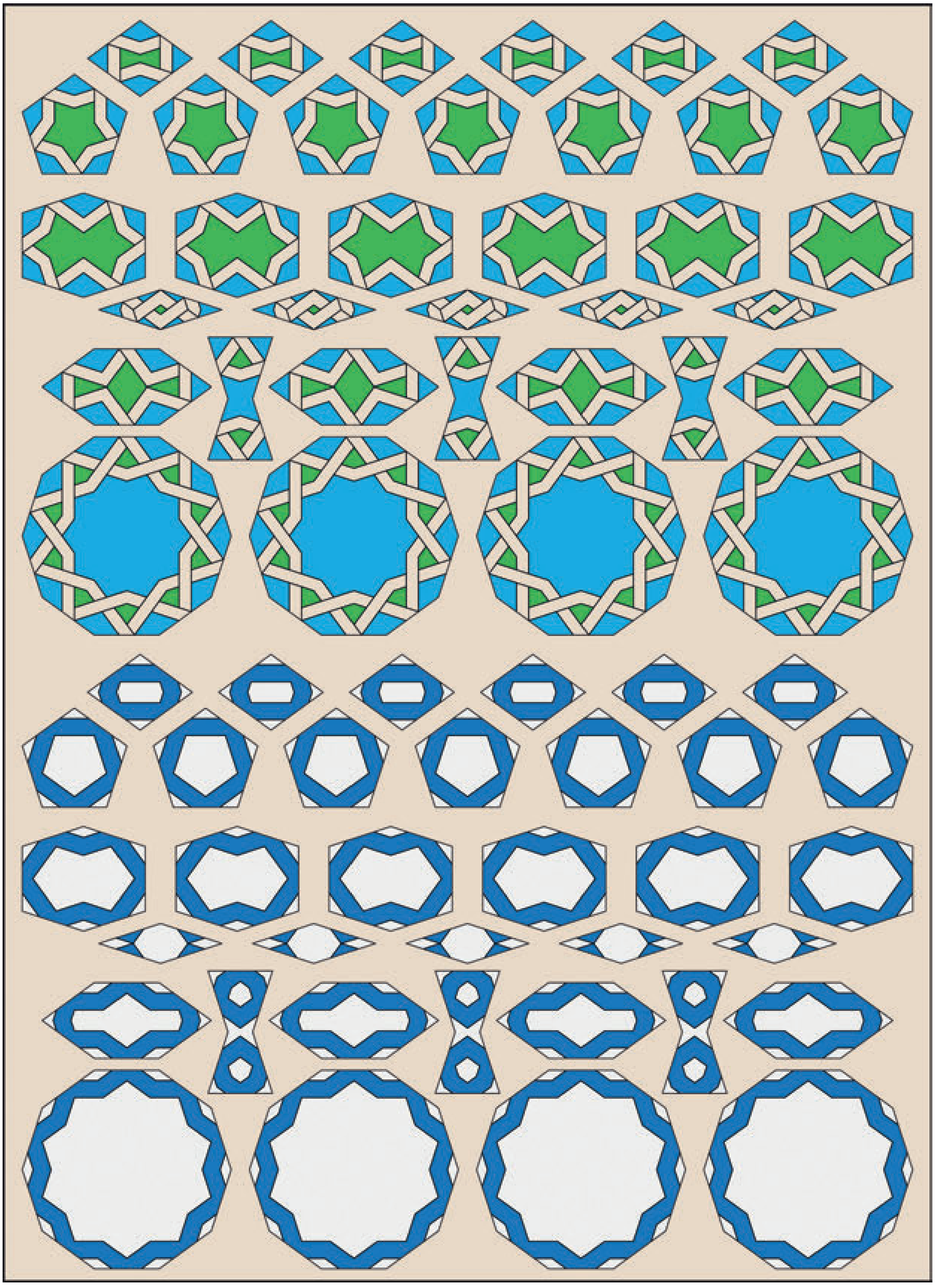
Activity 3B - The second activity in this section involves (1) photocopying the two sets of pre-decorated polygons; (2) carefully cutting them out; (3) assembling them into different design; (4) deciding upon a favorite design; and (5) gluing the favorite arrangement of polygons onto another piece of paper for permanence. Try multiple tessellations with each set before deciding upon a favorite.

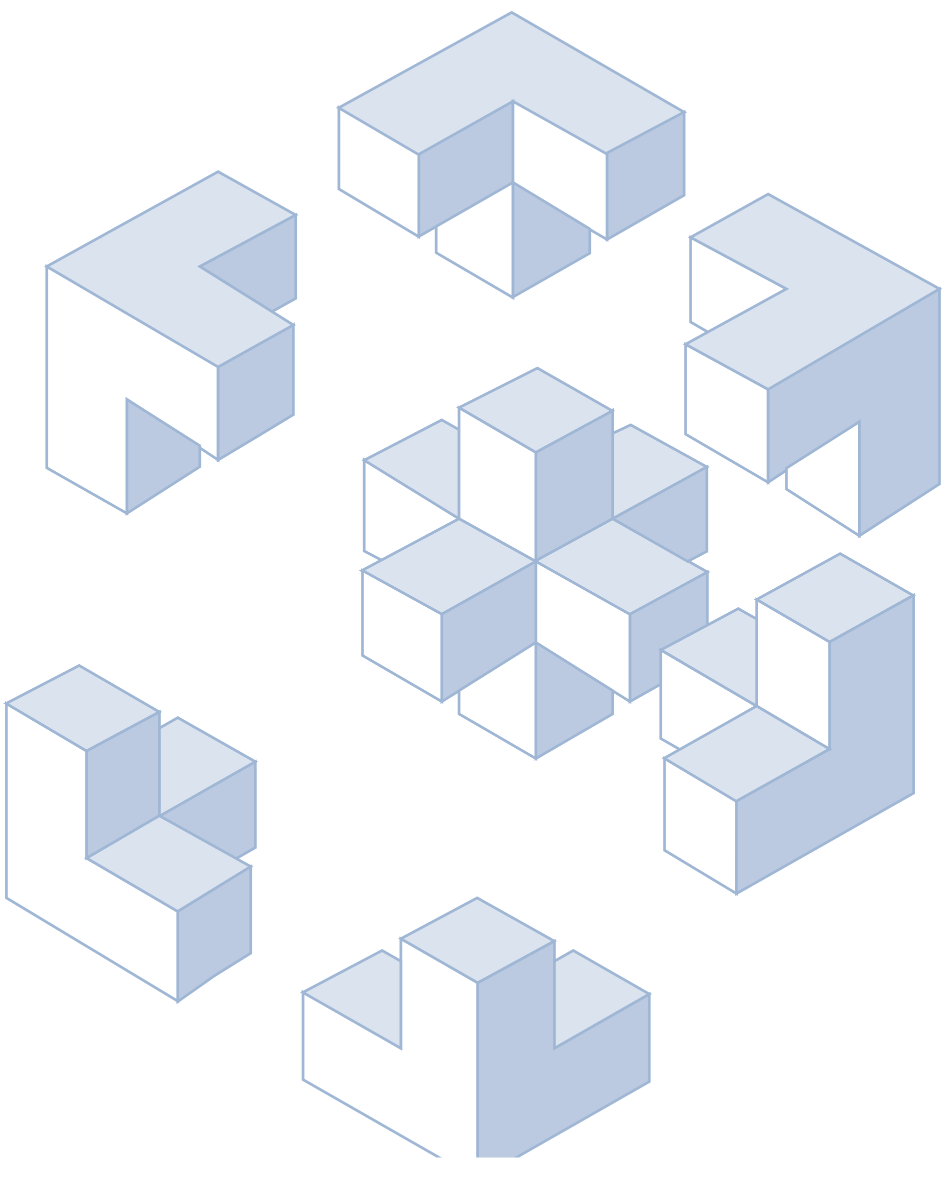


The following tessellation is the most widely used historically from the *5-fold system*. It works extremely well with each of the four pattern families.



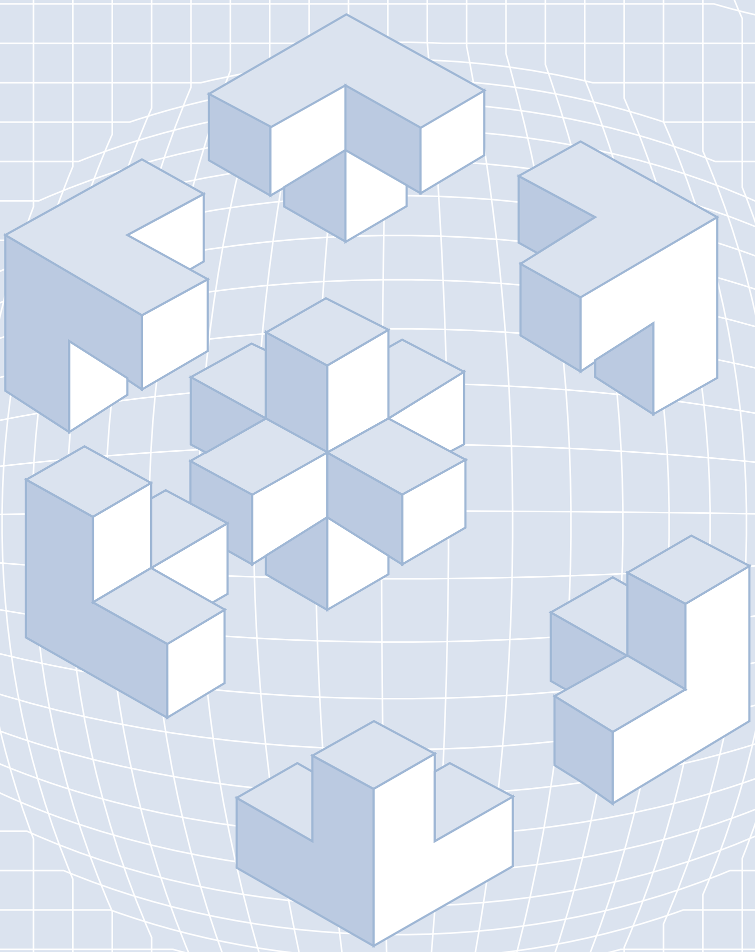
The upper set of polygons is decorated with pattern lines from the *median* family. The lower set is decorated with *obtuse* pattern lines. The tessellations that these polygonal elements will create can have rhombic repeat units, as well as rectangular and non-regular hexagonal repeat units. It is also possible to make tessellations and patterns with radial symmetry.





Jean-Marc Castera

Arabesques and Quasicrsystals



Visuality & Mathematics

Experiential Education of Mathematics through Visual Arts, Sciences and Playful Activities

Arabesques and Quasicrsystals

JEAN-MARC CASTERA is an artist working in the field of geometric art. Having a background in mathematics, he first taught at a university in Paris, then began working on 3D animation, and later on his own projects. Now his goal is to explore new paths in arabesque art, benefiting from connections with the theory of quasicrystals. Some practical applications, that were carried out with the well-knowed architect Norman Foster, can be seen in the Emirates.

Short bibliography:

"Arabesques", ACR editions, Paris 1996. English edition published in 1999.
"Géométrie douce", with Helene Jolis, Atelier 6 1/2, Paris, 1991.
"Flying Patterns". Source: http://castera.net/entrelacs/public/articles/Flying_Patterns.pdf
His recent papers are available on his website: castera.net

I propose two activities, using two set of tiles, related to the art of *zellij* (mosaics made from cut glazed ceramic tiles), and to the theory of quasicrystals. For each set of tiles you can make periodic or non-periodic patterns, and experiment with the property of self-similarity.

The first set of tiles belongs to the traditional repertory of the octagonal family of *zellij*, which is prevalent in the western part of the Islamic world, although the second set generate patterns with local 5-fold symmetry, more visible in the Persian style.

The related 2D quasicrystals are the Ammann patterns with the first set, and a system similar – but not identical – to the Penrose patterns with the second set.

Some Remarks about the Relationship between Octagonal Arabesque and Quasicrystals

We can talk about two levels in a *zellij* motif: the collective level, which means the structure of the whole panel, and the individual level considering the tiles individually.

The first connection with quasicrystals, which appeared when I was working on *zellij* patterns, concerns the collective level: the structure which I call the skeleton (first publication in "Géometrie douce", with Helene Jolis, Atelier 6½, Paris, 1991). The second connection appeared when working on a project in Abu Dhabi with Foster+Partners architects. It concerns the individual level: each tile can be broken into squares, rhombuses and half-squares, in such a way that when put together, in respect with the rules of the *zellij*, they compose a network similar to Ammann patterns.

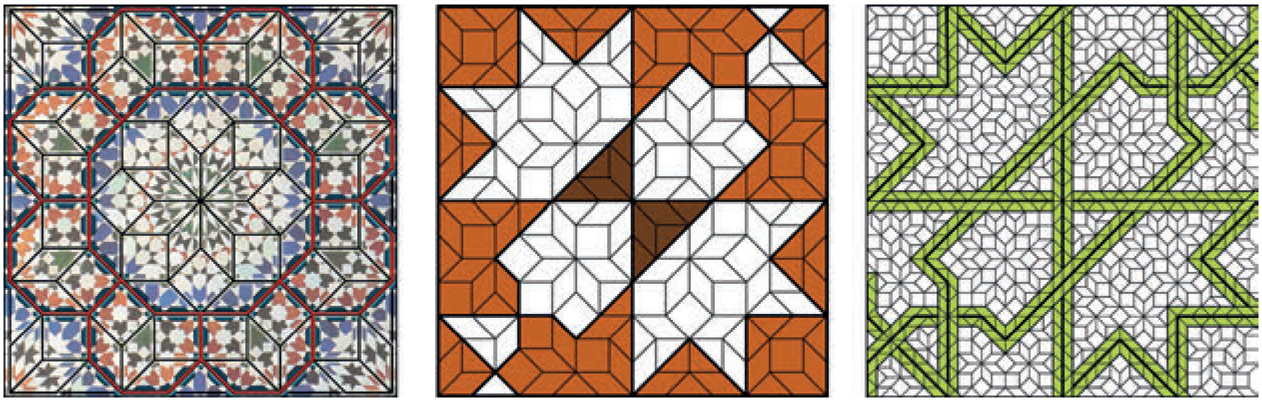


Figure 1.1. Three levels of connections between the octagonal family of zellij and the Ammann pattern: collective level (left), individual level simple (middle) or with interlaces (right).

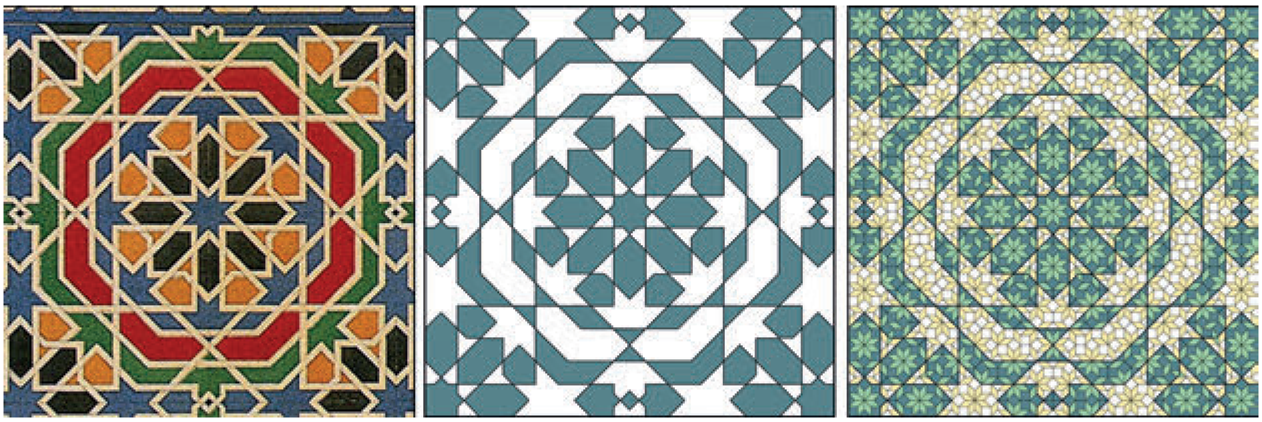


Figure 1.2. Example from Egypt (with obvious Western style influence). Left: actual pattern. Center: Drawing simple lines in place of the interlaces. Right: tiles decomposition, Ammann pattern.

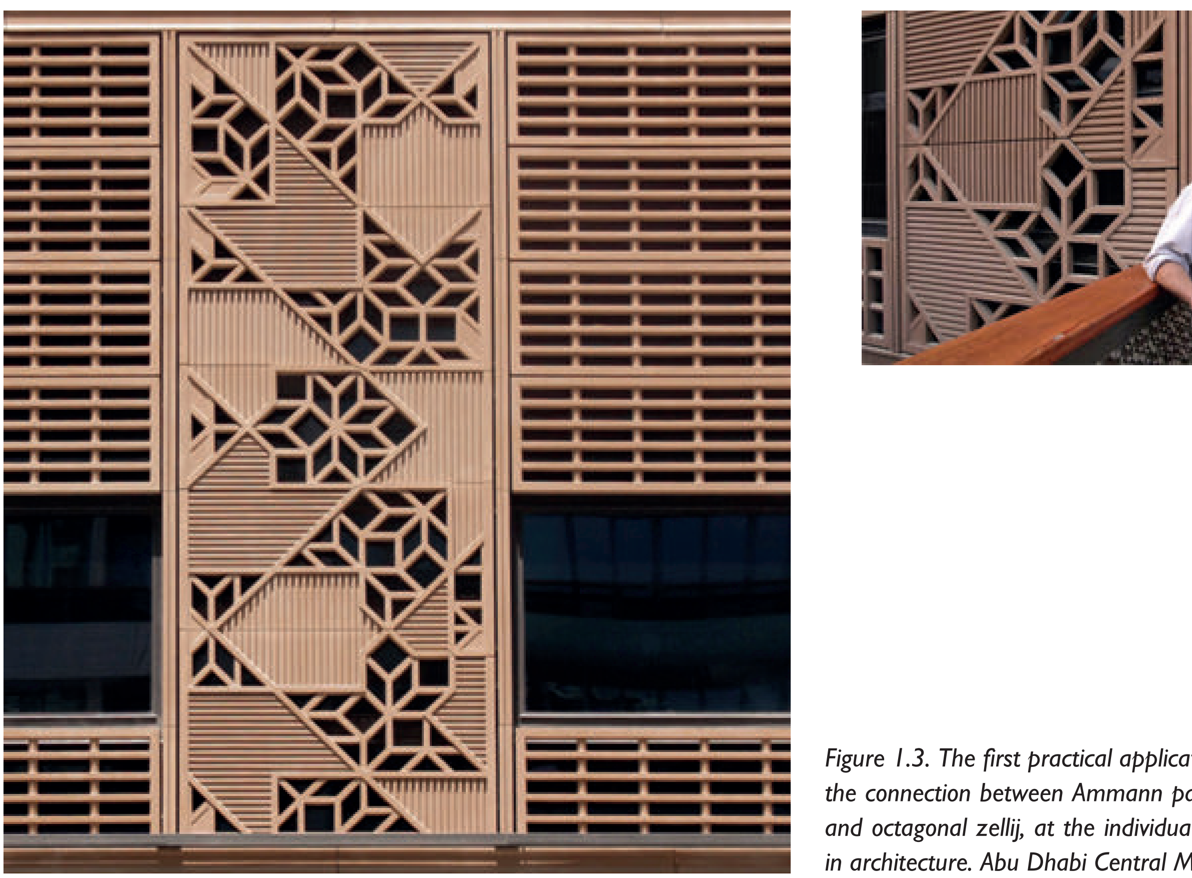


Figure 1.3. The first practical application of the connection between Ammann patterns and octagonal zellij, at the individual level, in architecture. Abu Dhabi Central Market.

Activity N°1 : Two Variations on the "Zellij Octagonal Puzzle" (Plate 1.1 ; plate 1.2)

Print and cut as many copies as possible (4 is ideal) of the set of tiles, then put the tiles together in order to make a pattern according to the rules of *zellij* (tiles connected side by side, continuity of the line in between, alternation of colours).

You can use the first model (Plate 1.1) or the second (Plate 1.2). You can also make a reverse side set, using one side from the first model, and the reverse side from the second. Use aerosol glue.

The shapes of each tile belong to the traditional repertory of octagonal *zellij*. The novelty is that each tile is decorated with squares, rhombuses and half-squares. When the tiles are correctly put together, each half-square naturally finds its mate, thus you can see a pattern made of squares and rhombuses superimposed on the *zellij* pattern. This pattern is similar to Ammann patterns, and can sometimes be considered as a part of an octagonal 2D quasicrystal.
Note: with these tiles you can make both periodic patterns or non-periodic patterns.

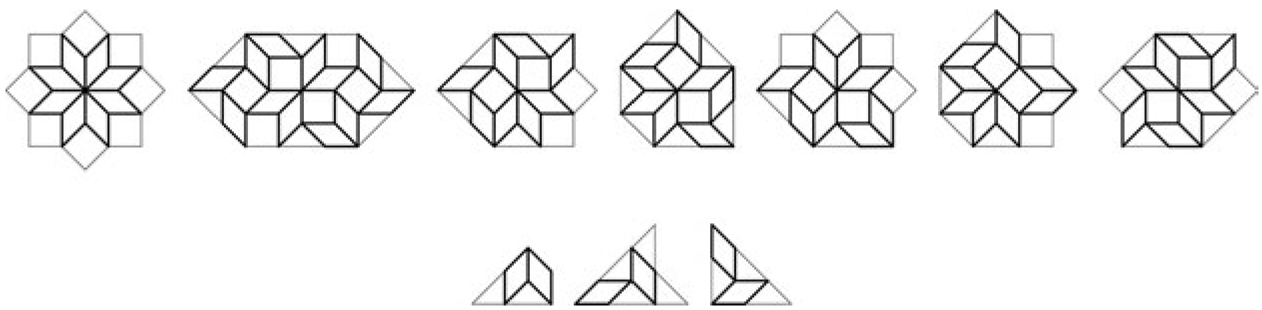


Figure 1.4. The different tiles, with decoration into squares, rhombuses and half-squares. These are the shapes you get from the plate 1.1.

Exercise n°1.1.

You can study the property of self-similarity with this set of tiles. The figure bellow shows the way in which each tile can be made of tiles belonging to the same set, on a smaller scale (although along the edges the tiles in red are truncated).

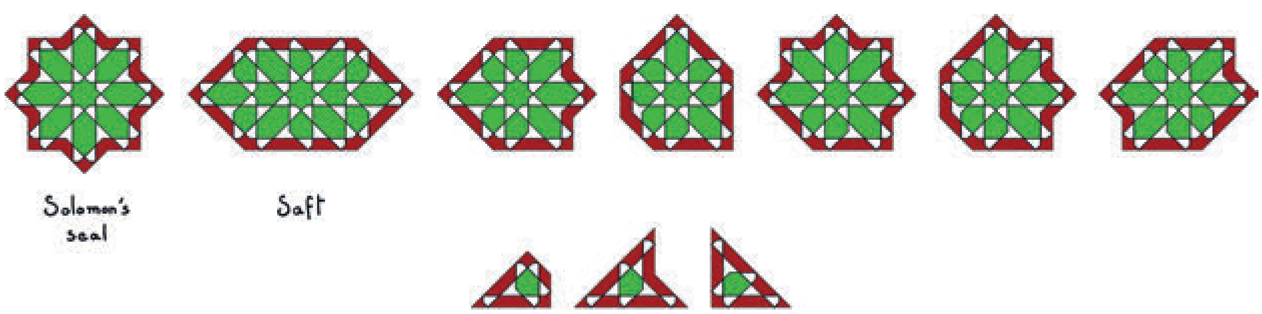


Figure 1.5. Self similarity of the "zellij" tiles.

1. What is the ratio between the two scales?
2. Application: in an opposite way each tile, for example the "Saft", can be constructed on a larger scale (see fig. 3.2).

Cut the tiles of *Plate 1.1.* and *Plate 1.2.* along the median lines in red. On *Plate 1.2.* the second page, the secondary level of *zellij* tiles and the secondary level of octagonal pattern have been drawn.

When assembling the tiles, pay attention to how the colours alternate.

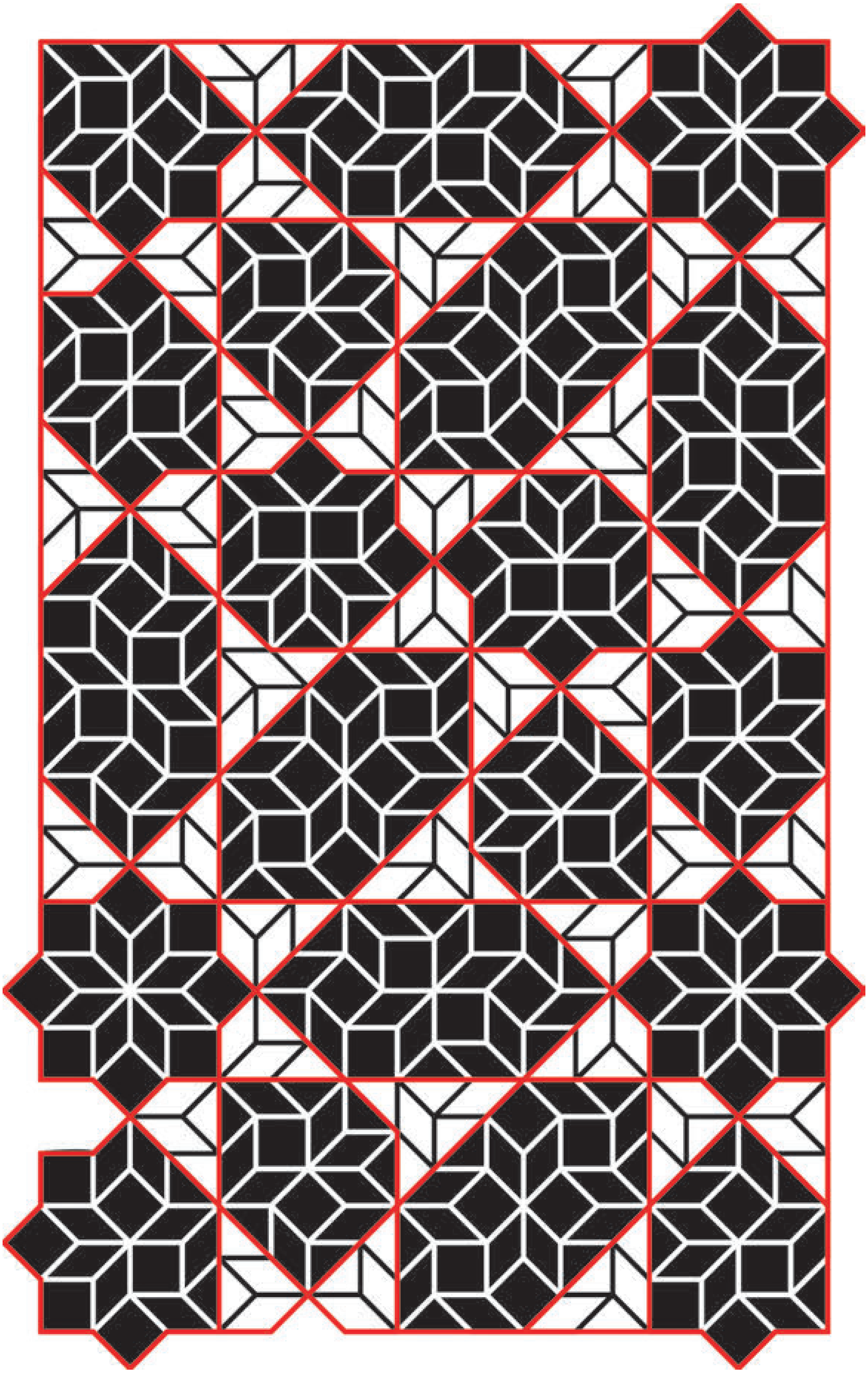


Plate 1.1. Octagonal family of zellij patterns and Ammann patterns. Cut the tiles along the red line.

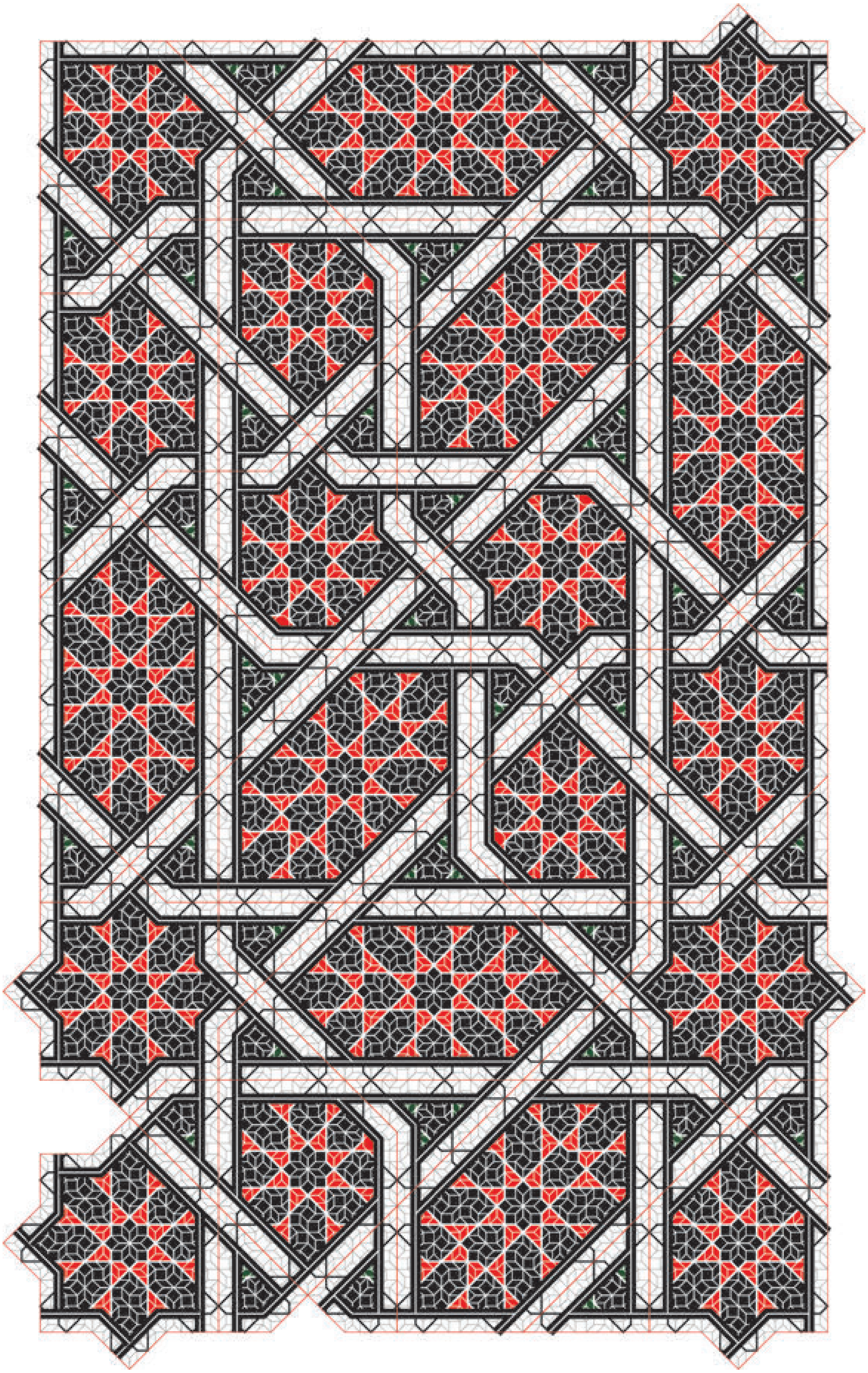


Plate 1.2. Octagonal family of zellij patterns, self similarity. Cut the tiles along the red line (same thing as for the previous page).

Activity N°2: The X-Puzzle

X-puzzle is made of a set of only two different tiles (the "X-tiles") decorated in a simple way, using only two lines to make an X (the "X-lines").

The length of each side is equal, and the angles are such that 10 small tiles can make a 10-pointed star, and 5 large ones can make a 5-pointed star.

These tiles are the same as the ones in Penrose patterns, but the rules are not the same: although Penrose matching rules avoid making periodic, the patterns made from the X-Puzzle can be periodic or non-periodic.

The idea of the X-tiles arose when I was looking for a kind of morphogenesis of the pentagonal family of traditional patterns (especially in Persian style, see: Jean-Marc Castera, "Flying Patterns", source: http://castera.net/entrelacs/public/articles/Flying_Patterns.pdf). Those simple shapes were the very first step in my system, eventually it transpired that, despite their uncomplicated pattern, they were powerful in their simplicity: they can be used to generate a sophisticated family of patterns in that style (I do not claim, however, that this method was used by craftsmen in the historic period). The most fascinating to see here is the relationship between complexity and simplicity.

The matching rules are obvious: the shapes have to be put together side by side respecting the continuity of the X-lines.

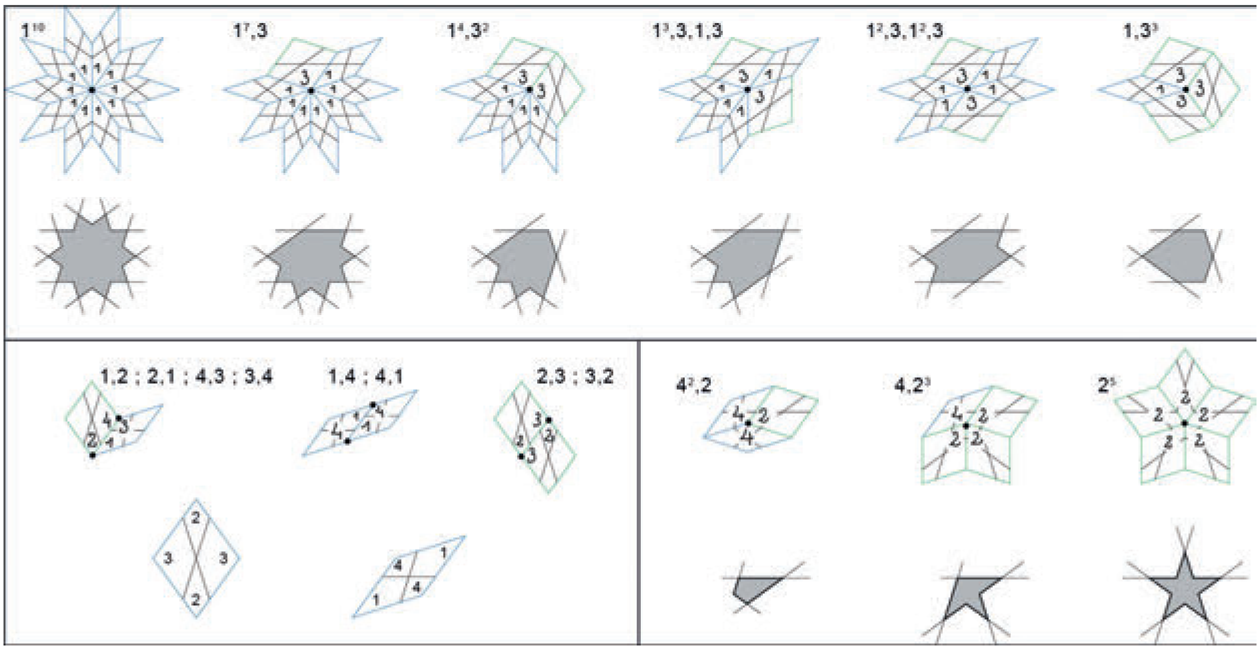


Figure 2.1. Top and bottom-right: All the correct arrangement of tiles around a vertex, and the corresponding "zellij" shapes drawn by the X-lines. Bottom-left: Incorrect arrangements.
The numbers indicate the angles around the vertex. "1" is for 36°, "2" for 72°, etc.

You can see that a finite number of "*zellij-like*" shapes can be drawn by the X lines.

Different Kinds of Compositions

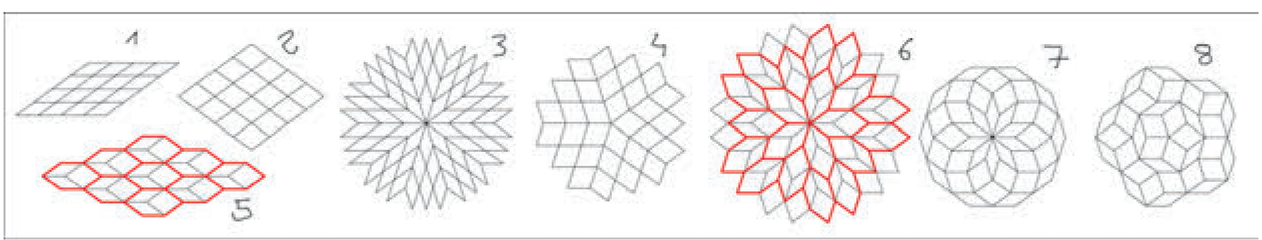


Figure 2.2. Compositions with the two kind of rhombuses. (1,2): periodic, (5): hexagonal => periodic; (3,4,6): radial; (7,8): pentagonal symmetry => non periodic;
This works with simple rhombuses. But with X-lines decoration, compositions (1, 2, 3, 4, 8) are incorrect.
On the other hand, we can also play freely with the tiles.

Remark: "Irreductible Holes"

Sometimes you can get stuck when a space limited by some tiles cannot be filled. The natural reaction is to change the pattern in a way to avoid that situation. But don't be afraid of the void: you can always find a way to go around it so that the pattern can continue. In the end what you get is a pattern... With one or more holes.

A hole is irreductible when it is impossible to put any X-tile in it respecting the continuity of the X-lines (considering the surrounding X-tiles).

Figure 2.3. Different kinds of irreductible holes.

Remark: The X-tiles are decorated using large white lines, one crossing over the other. The amusing thing is the way the interlaces naturally run on the pattern in respect to the alternation: above, below, above...

Exercise 2.1: What is the relationship between the angles in each tile and the ones between the X-lines?

Exercise 2.2: The holes.
How many different shapes of "irreductibles holes" can exist? The answer is not obvious!

Exercise 2.3: Self-similarity.
Can you define a self-similarity property for each of the two tiles? That is, a way to compose the same shapes of rhombuses, but on a larger scale with the X-tiles.
In such case, what is the self-similarity ratio?

Exercise 2.4: Can you imagine a similar puzzle from the octagonal square/rhombus system?

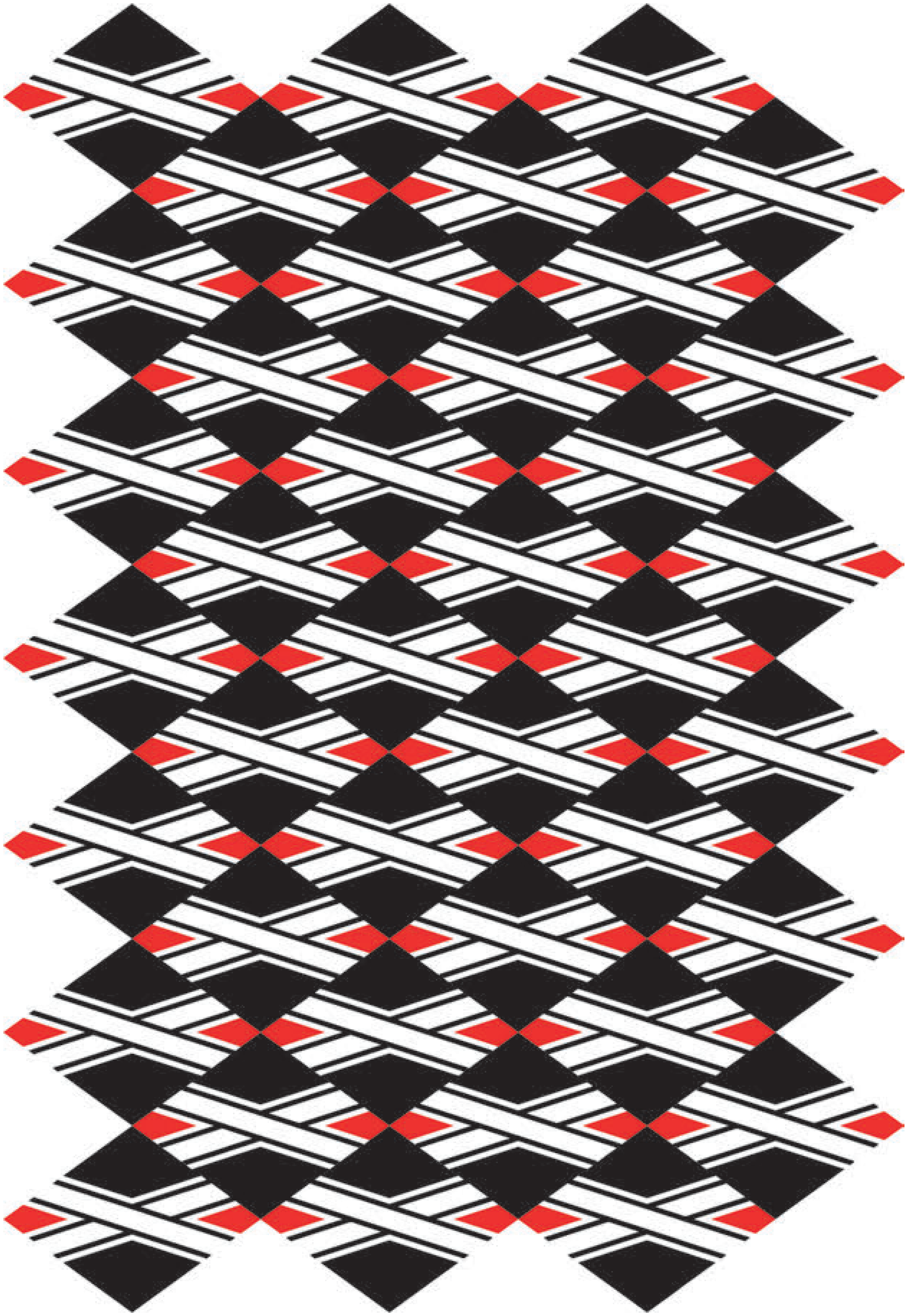


Plate 2.1. Cut the rhombuses (each tile is the same as the separate one at the bottom).

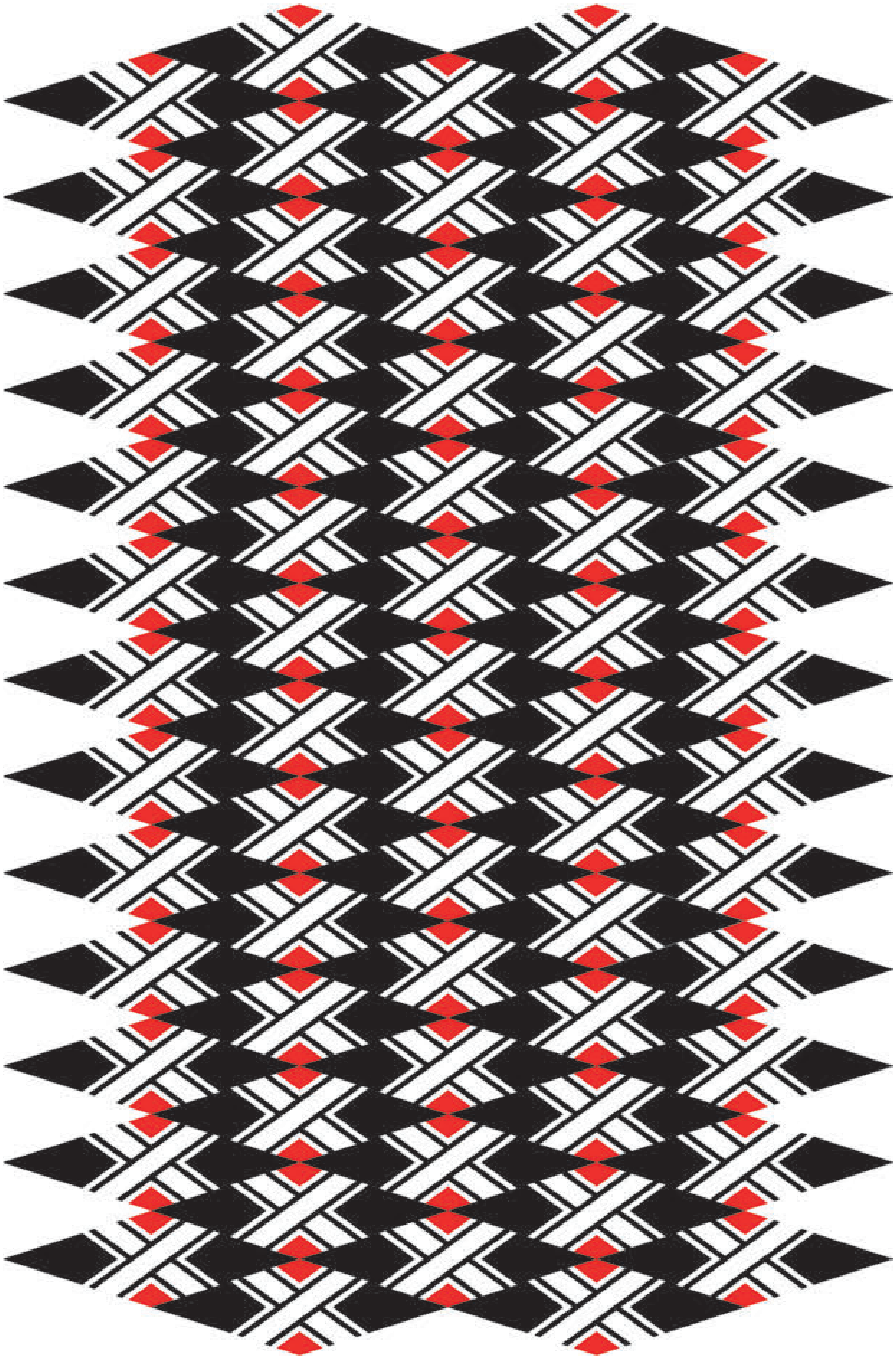


Plate 2.2. Cut the rhombuses (each tile is the same as the separate one at the bottom).

Figure 3.1. Children playing with two versions of the "X-Puzzle" at a math festival in Paris.

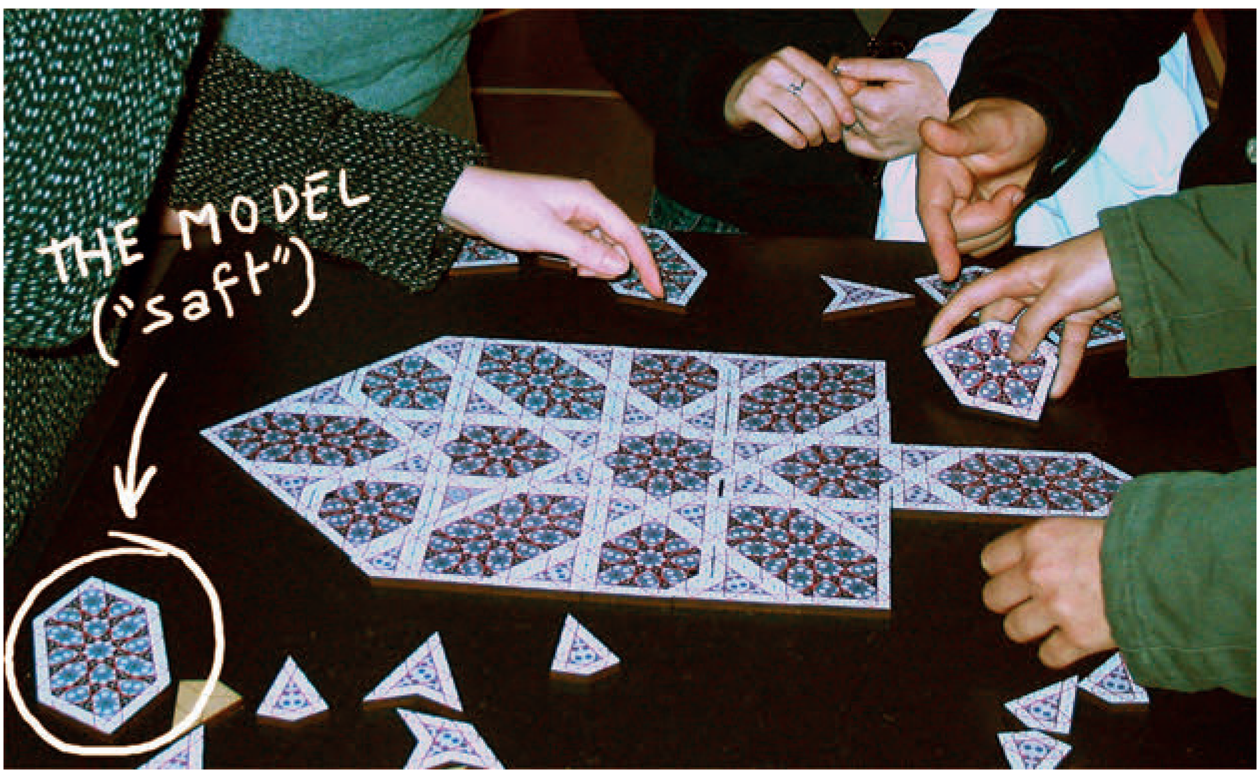


Figure 3.2. A way to use the octagonal tiles: People making a big "saft" by "deflation" at the Palais de la découverte (Science museum) in Paris.

Figure 3.3. An example of using traditional art: a door at űç Serefeli mosque, Edirne, Turkey.
Left: The original door; Middle: decomposition into rhombuses. Right: reconstruction with the X-tiles.

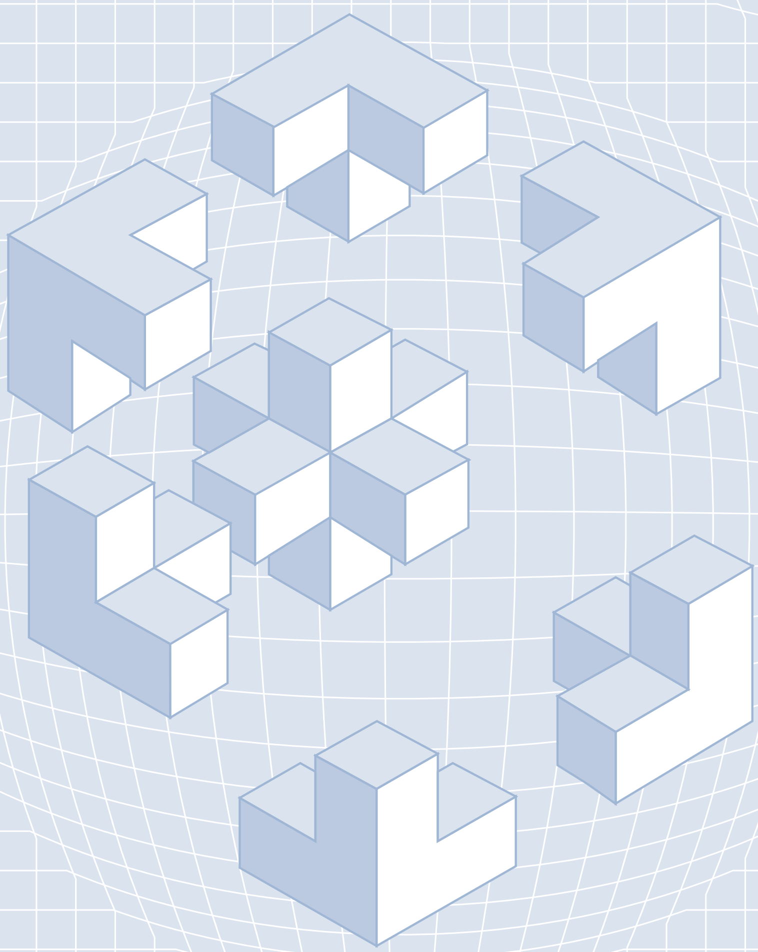
Figure 3.4. An other door at the same mosque (űç Serefeli mosque, Edirne, Turkey).
Left : The original door; Middle: decomposition into rhombuses. Right: reconstruction with the X-tiles.

Visuality & Mathematics

Experiential Education of Mathematics through Visual Arts, Sciences and Playful Activities

Reza Sarhangi

A Geometric Activity in Patterning Platonic and Kepler-Poinsot Solids Based on the Persian Interlocking Designs



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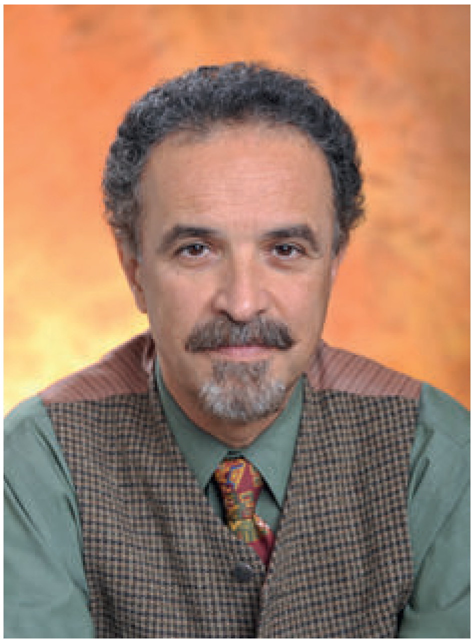

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Related articles:

Reza Sarhangi (2014), *Decorating Regular Polyhedra Using Historical Interlocking Star Polygonal Patterns – A Mathematics and Art Case Study,* Refereed Conference Proceedings, Bridges 2014 - Mathematical Connections in Art, Music, and Science, Seoul, Korea.
Web-access: http://archive.bridgesmathart.org/2014/bridges2014-243.pdf

Reza Sarhangi (2012), *Interlocking Star Polygons in Persian Architecture: The Special Case of the Decagram in Mosaic Designs,* Nexus Network Journal, Architecture and Mathematics, Volume 14, Number 2, Birkhäuser Verlag, Springer, Heidelberg, Germany, pp. 345-372.

Reza Sarhangi (2010), *Making Modules for Mosaic Designs,* The Journal of the Symmetrion (Symmetry: Culture and Science), Michel Petitjean, Editor-in-Chief of Symmetry, MTI, INSERM UMR-S 973, University of Paris, France, Vol. 21, No. 4, PP. 381-392.

Introduction

The *regular polyhedra* are highly organized structures that possess the greatest possible symmetry among all polyhedra, which makes them aesthetically pleasing. These solids have connected numerous disciplines including astronomy, philosophy, and art through the centuries. The five that are convex are the *Platonic* solids and the four that are not convex are the *Kepler-Poinsot* solids. They admit the properties that for each (a) all faces are congruent regular polygons (convex or non-convex), and (b) the arrangements of polygons about the vertices are all alike.

Platonic solids were known to humans much earlier than the time of Plato. On carved stones (dated approximately 2000 BCE) that have been discovered in Scotland, some are carved with lines corresponding to the edges of regular polyhedra. Icosahedral dice were used by the ancient Egyptians. There are many small bronze dodecahedra that were discovered from the time of Romans of the second to fourth century that are decorated with spheroids at each vertex and have circular holes in each face. Evidence shows that Pythagoreans knew about the regular solids of cube, tetrahedron, and dodecahedron. A later Greek mathematician, Theatetus (415 - 369 BCE) has been credited for developing a general theory of regular polyhedra and adding the octahedron and icosahedron to solids that were known earlier. The name Platonic solids for regular polyhedra comes from the Greek philosopher Plato (427 - 347 BCE), who associated them with the "elements" and the cosmos in his book *Timaeus*. "Elements," in ancient beliefs, were the four objects that constructed the physical world; these elements are fire, air, earth, and water.

There are four more regular polyhedra that are not convex. Johannes Kepler (1571-1630 CE) discovered two of them, the *small stellated dodecahedron* and the *great stellated dodecahedron*. Later, Lovis Poinsot (1777 – 1859 CE) completed the work by finding the other two non-convex regular polyhedra of the *great icosahedron*, and the *great dodecahedron.* For a comprehensive treatment and for references to the extensive literature on solids one may refer to the online resource *Virtual Polyhedra* – The Encyclopedia of Polyhedra [3].

The Schläfli Symbols (n, m) in the following table present the relationships between the n-gon, as the face of the regular polyhedron, and m, which is the number of faces around a vertex for that polyhedron. By a regular pentagram in this table, we mean a 5/2 star polygon, which is the regular non-convex pentagon. Figure 1 demonstrates the regular solids.

Polyhedron	Schläfli Symbol	Complete Face	Visible Face
Tetrahedron	(3, 3)	Equilateral Triangle	Equilateral Triangle
Octahedron	(3, 4)	Equilateral Triangle	Equilateral Triangle
Icosahedron	(3, 5)	Equilateral Triangle	Equilateral Triangle
Hexahedron (Cube)	(4, 3)	Square	Square
Dodecahedron	(5, 3)	Regular Pentagon	Regular Pentagon
Small Stellated Dodecahedron	(5/2, 5)	Regular Pentagram	Golden Triangle
Great Stellated Dodecahedron	(5/2, 3)	Regular Pentagram	Golden Triangle
Great Icosahedron	(3, 5/2)	Equilateral Triangle	Two triangles in Fig 12
Great Dodecahedron	(5, 5/2)	Regular Pentagon	Obtuse Golden Triangle

Table1: *The regular polyhedra*

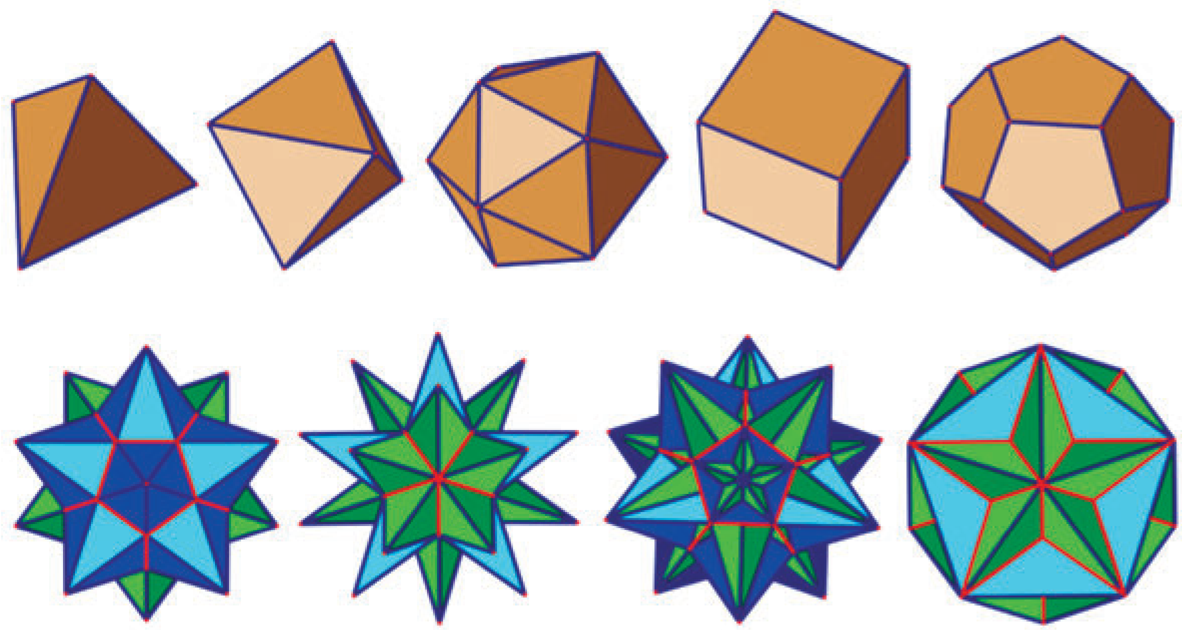


Figure 1: *Top – tetrahedron, octahedron, icosahedron, hexahedron,* and *dodecahedron. Bottom – small stellated dodecahedron, great stellated dodecahedron, great icosahedron, and great dodecahedron*

The goal of this article is to present methods for the decoration of the regular polyhedra using Persian *girih* patterns and to provide instructions for making these polyhedra from paper. *Girih* (knot in Persian) refers to a (usually) rectangular region consisting of a fundamental region with bilateral or rotational symmetry, for a pattern that includes the nodal points of the web-like geometric grid system and construction lines for generating the pattern.

2 Some Examples in Patterning Solids

There are numerous interesting examples of patterning regular solids. The left image in Figure 2 exhibits the *Screened Icosahedron* created by artist Phil Webster from Pittsfield, Massachusetts. The artwork was presented at the 2013 Bridges Conference Art Exhibition, Enschede, the Netherlands [2]. The right image in Figure 2, the ornamented great dodecahedron created by Richard Kallweit, an artist from Bethany, Connecticut, was presented at the 2014 Joint Mathematics Meeting Art Exhibition, Baltimore, Maryland, USA [2].

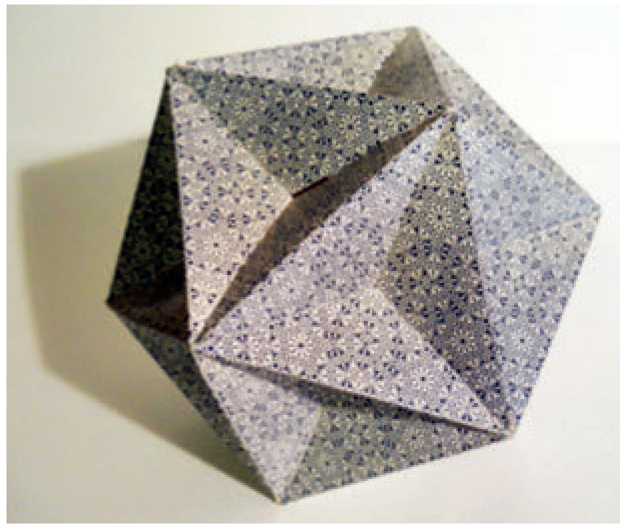


Figure 2: *Screened Icosahedron and embellished Great Dodecahedron*

Captured Worlds by artist Dick Termes (http://termespheres.com) is a set of Platonic Solids that are decorated with fanciful scenes rendered in six-point perspective, which allows an entire three-dimensional surrounding to be projected onto the polyhedra (left in Figure 3). B.G. Thomas and M.A. Hann from the School of Design, University of Leeds, United Kingdom, used the projections from duals to the surface of the Platonic solids, in particular the dodecahedron, in order to decorate the faces of the polyhedra (right in Figure 3) [10].

Figure 3: *Captured Worlds by artist Dick Termes, and an example of the projection of the pattern on the cube to the dodecahedron by B.G. Thomas and M.A. Hann*

During a workshop in the 2010 Bridges Pécs Conference, Hungary, E.B. Meenan of the School of Education and B.G. Thomas of the School of Design, University of Leeds, UK, guided their workshop participants to a process of creating Escher-type tessellations. Then they used presented ideas to extend the workshop into three-dimensions with pull-up Platonic solids constructions that were patterned with Escher's designs [6].

3 Historical Patterns for Embellishment of Solids

It is important to note that since a pattern on one face of an ornamented solid should appear on all faces identically, all the pattern lines should be in complete coordination and harmony with each other in such a way that they can continue from one face to another without any ending or interruption.

There are only a limited number of scrolls (*tumār*) and booklets (*daftar*) from the past that recorded patterns and designs for the decorations of the surfaces of buildings, or as geometric experimentations of interlocking star-polygon patterns. But in general, such designs come with no instruction about the steps of the geometric construction using traditional tools of compass and straightedge or any other tools.

3.1. Patterning Platonic Solids. To decorate regular polyhedra using historical and traditional interlocking star polygonal patterns, one needs to search the documents for ornamented polygons of the equilateral triangle, square, and regular pentagon. Beginning with the dodecahedron and its face, the regular pentagon, the author searched most of the available old documents for a decorated pentagon with the following specifications:

a, The center of the pentagon coincides with the center of a *k/l* star polygon that covers the central region of the pentagon.
b, The vertices coincide with the centers of the same or other *k/l* star polygons that cover corners of the pentagon.
c, Some segments connect star polygons together in a harmonious way to generate a single design.

Mathematically speaking, if all the *k/l* star polygons that are used for ornamenting the pentagon are identical, then since each interior angle of the regular pentagon is $\frac{(5-2)\pi}{5} = 108°$, and since each vertex of the dodecahedron includes three copies of the pentagon, on a successful patterning, a type of star polygon will appear on each vertex that covers 324°. This means the *k/ l* star polygon on the center, which is an *n*-leaved rose that covers 360°, should be constructed in a way that the number of degrees in each leaf divides both 360 and 324, as does their difference, 36. Therefore k should be equal to $5i$, $i \in \mathbb{N}$. Hence, star polygons such as 5/*l*, 10/*l*, 15/*l* and so on will provide proper central designs for the pentagon (that will create concave star polygons of 3/*l*, 6/*l*, 9/*l* and … on the corners of the dodecahedron).

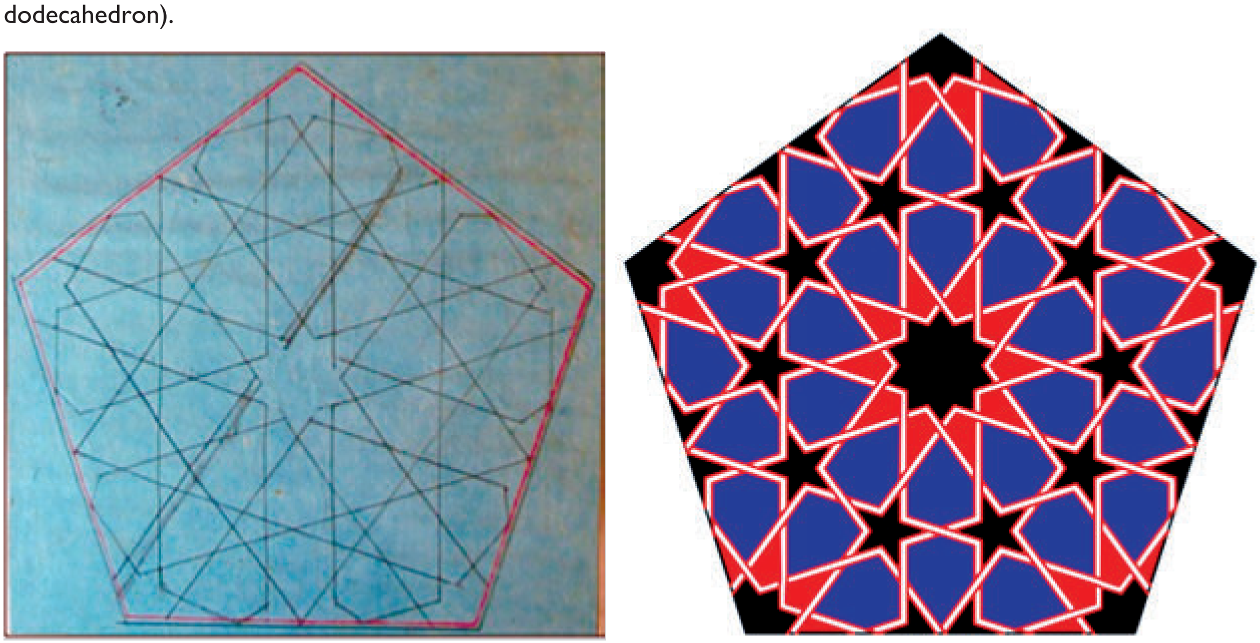


Figure 4: *The decorated pentagon from the Mirza Akbar collection, and the ornamented pentagon created by the author based on the mathematics in the Mirza Akbar ornatemented pentagon*

There are many sources, including buildings of the past, that we can search for Persian traditional patterns and we may find many examples. One source of interest for finding such a pattern was the *Mirza Akbar Collection,* which is housed at the Victoria & Albert Museum, London. This collection consists of two architectural scrolls along with more than fifty designs that are mounted on cardboard. The collection was originally purchased for the South Kensington Museum (the precursor of the Victoria & Albert Museum) by Sir Caspar Purdon Clarke, Director of the Art Museum (Division of the Victoria and Albert Museum) 1896–1905 in Tehran, Iran, in 1876. Purdon Clarke purchased them after the death of Mirza Akbar Khan who had been the Persian state architect of this period [7].

The left image in Figure 4 is from the Mirza Akbar collection. As is seen, the constructed lines in this image are not accurate and the pattern looks like a draft. Nevertheless, the design and the 10/3 star polygon at the center satisfy the aforementioned constraints. The image on the right in Figure 4, which was created by the author, using the Geometer's Sketchpad program, illustrates the same pattern but includes interwoven straps, which changes the symmetry group of the pattern from the dihedral group of order 10, D_5, to the cyclic group of order 5, C_5.

Obviously, for the geometric construction of the pattern, there were no instructions in the Mirza Akbar collection, so it was necessary to analyze it mathematically, to discover the construction steps.

One should notice that the radius of the circumscribed circle of the pentagon in Figure 5, *OA*, which is the distance from the center of the pentagon to a vertex, is twice the radius of the circle that is the basis for the 10/3 star polygon at the center (*AM* = *MO* in the top left image in Figure 5). The reason for this is that the two 10/3 star polygons, one at the center *O* and the other at the vertex *A*, are each others reflections under the tangent to the circle at point *M* (see the middle bottom image in Figure 5 that also includes a tangent to the circle at point *N* that is necessary to be used as the reflection line, to complete the star). By following images from the top left to the bottom right a person may complete the design properly.

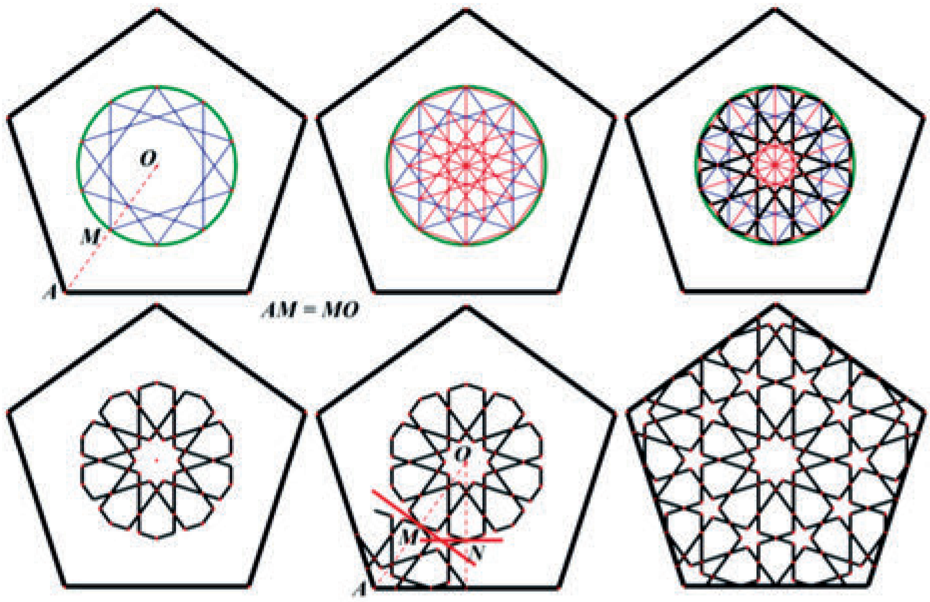


Figure 5: *The steps for the geometric composition of the Mirza Akbar ornamented pentagon*

The photographs in Figure 6 are from a workshop conducted by the author that was presented at the Istanbul Design Center. The workshop was a part of a conference on geometric patterns in Islamic art that was scheduled during 23rd-29th September 2013 in Istanbul, Turkey. The workshop included the construction of the dodecahedron using the Mirza Akbar ornamented pentagon.

Figure 6: *Photographs from a workshop in the Istanbul Design Center in Turkey*

The next selected polygon for embellishment was the square. It was not difficult to find a decorated square in the Mirza Akbar collection. However, the design, as can be noted in the left image of Figure 7, was a very rough draft with no accuracy on any part of the design, showing only the type of polygons that constituted the structure, but nothing to assist a designer to determine the steps of the geometric constructions. Searching a book by J. Bourgoin [1], plate 118 in this book exhibits the same structure, but the proportions are slighly different from the sketch in the Mirza Akbar collections. This book consists of 190 geometric construction plates that appeared in the French edition, *Les Eléments de l'art: le trait des entrelacs, Firmin-Didot et Cie*, Paris, 1879. The book does not provide step-by-step instructions for the geometric constructions. Nevertheless, there are underlying circles and segments using thin dashed lines that are instrumental for forming such instructions.

To form the instructions for the pattern illustrated in Figure 8, plate 118 was used but a few steps were changed, to be more in tune with the traditional approaches to complete the ornamented square in Figure 7 (right image).

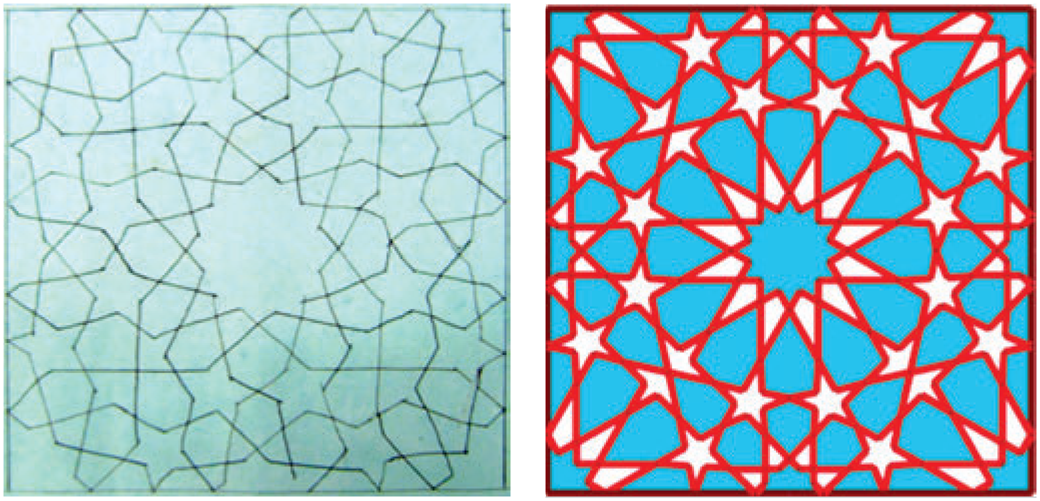
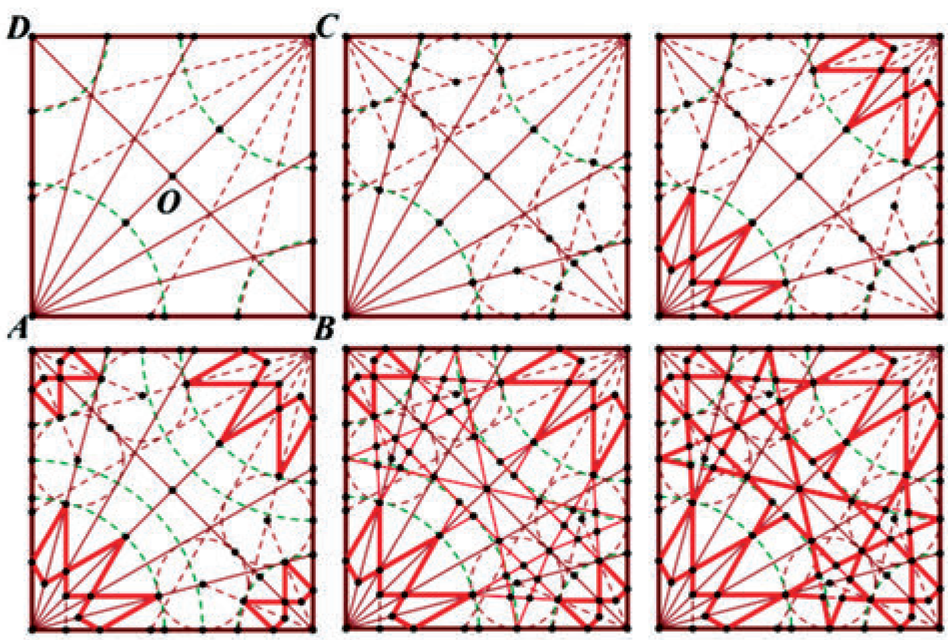


Figure 7: *The decorated square from the Mirza Akbar collection, and the ornamented pentagon created by the author based on the types of tiles that constituted the Mirza Akbar square*

The construction approach, the *radial grid method*, follows the steps that the mosaic designer Maheroannaqsh suggested for another pattern in his book [5].

Divide the right angle ∠*A* into six congruent angles by creating five rays that emanate from *A*. Choose an arbitrary point *C* on the third ray, counter-clockwise, and drop perpendiculars from *C* to the sides of angle ∠*A*. This results in the square *ABCD,* along with the five segments inside this rectangle, each with one endpoint at *A,* whose other endpoints are the intersections of the five rays with the two sides of *BC* and *CD* of square *ABCD*. Consider *C* and the dashed segments as the 180° rotational symmetry of *A* and the five radial segments under center *O*. Make a quarter of a circle with center at *A* and radius equal to 1/3 of *AC*. We repeat all these and the future construction steps for *C*. Two quarter circles can be constructed at *B* and *D* with radius congruent to the distance from *B* to the intersection of the first ray and *BC*. The two right angles at *B* and *D* are divided each into four congruent angles. The intersections of some new rays emanating from *B* and *D* and the previous constructed rays emanating from *A* and *C*, as are illustrated in the second top image, are the centers of the circles that are tangent to the sides of the square *ABCD* and some rays. Dropping a perpendicular from the intersection of the quarter circle with center *A* and the fifth ray emanating from *A* to *AB* will result in finding some new points on other rays for the construction of a quarter of a 12-leaved star with center at *A* (the top right image). With a similar approach one can construct one quarter of an 8-leaved star with center at *B*, as is illustrated in the first bottom image. As is illustrated in the bottom middle image, some segments are constructed that connect the four stars in the four corners. Morover, some other segments connect the intersections of the four small circles with the sides of *ABCD*. The last part of the construction, as is seen in the bottom right imgae, is to complete the pattern. The right image in Figure 7 is the result of the reflection of this decorated square under the two sides of *AB* and *AD*.

Figure 8: *The steps for the geometric construction of the Mirza Akbar ornamented square*



For patterning the tetrahedron, octahedron, and icosahedron, one needs an ornamented equilateral triangle. To follow the types of patterning in the pentagon and the square, a 12/4 star polygon was inscribed inside the given equailateral triangle. Then similar to the steps in Figure 5, steps were taken to ornament the triangle (Figure 9).

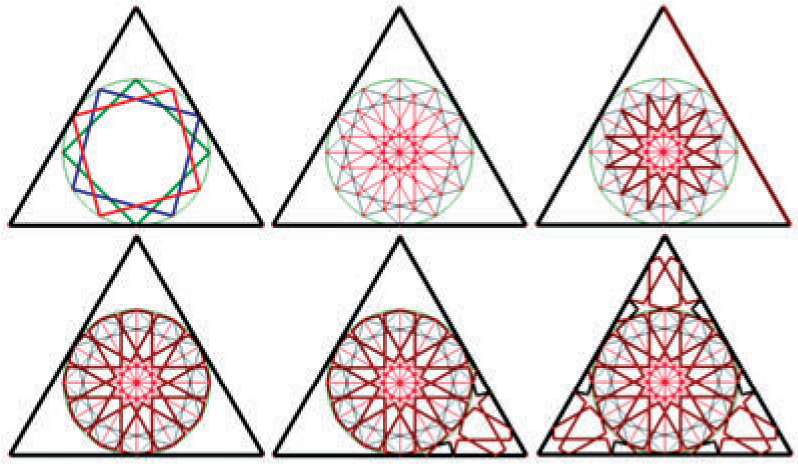


Figure 9: *The steps for the geometric construction of the ornamented triangle*

3.2. Patterning Kepler-Poinsot Solids. The faces of these solids are the regular pentagram 5/2 in Figure 10, regular pentagon, and equilateral triangle. However, faces cross each other and therefore, the physical models have visible faces that are different from the actual faces. The visible faces of the physical models of (5/2, 5), (5/2, 3), and (5, 5/2) are either the *golden triangle* (*DABD* in Figure 10, an isosceles triangle with angles 72, 72, and 36 degrees) or an *obtuse golden triangle* (*DBDC* or *DADE* in Figure 10, an isosceles triangle with angles 36, 36, and 108 degrees). Therefore, for patterning the above three Kepler-Poinsot polyhedra, we need to ornament these two triangles.

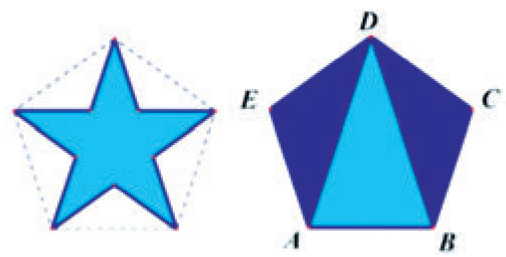


Figure 10: *The regular pentagram 5/2, and the pentagon divided into the golden triangle and obtuse golden triangle*

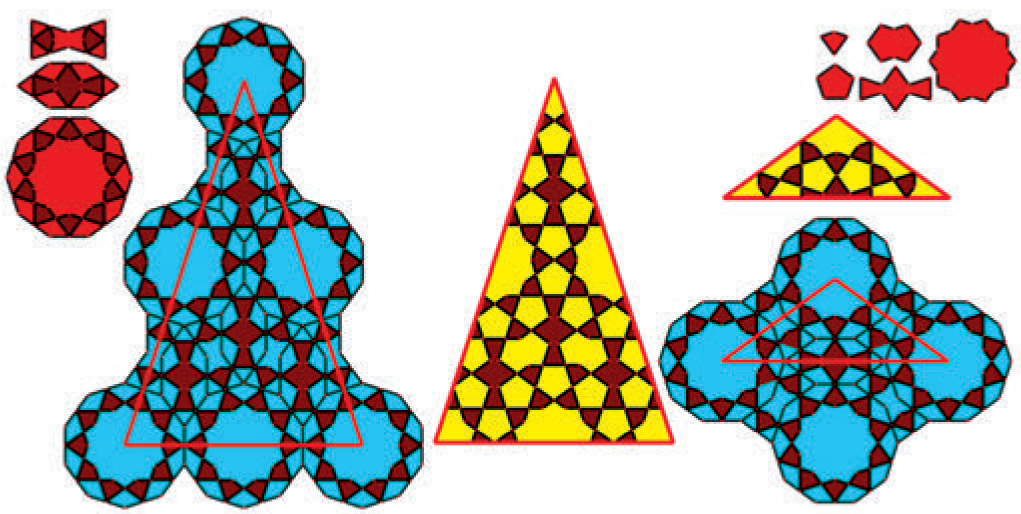


Figure 11: *Ornamenting the golden triangle and the obtuse golden triangle using girih tiles*

For this, the girih tile modularity method presented in [4] was used. In [4] the authors proposed the possibility of the use of a set of tiles, called girih tiles (top left corner of Figure 11) by the medieval craftsmen, for the preliminary composition of the underlying pattern. The pattern then would be covered by the glazed *sâzeh* tiles (top right corner of Figure 11) in the last stage. On the top right image in Figure 11, we see the three girih tiles, which are used to compose the underlying pattern on the two triangles of the golden triangle and the obtuse golden triangle. After finding the pattern, all line segments that constitute the girih tiles are discarded (see the two triangles in Figure 11). Then the *sâzeh* tiles that are presented on the top right corner are used to cover the surface area. For a comprehensive explanation of this and other modularity methods the interested reader is referred to [8].

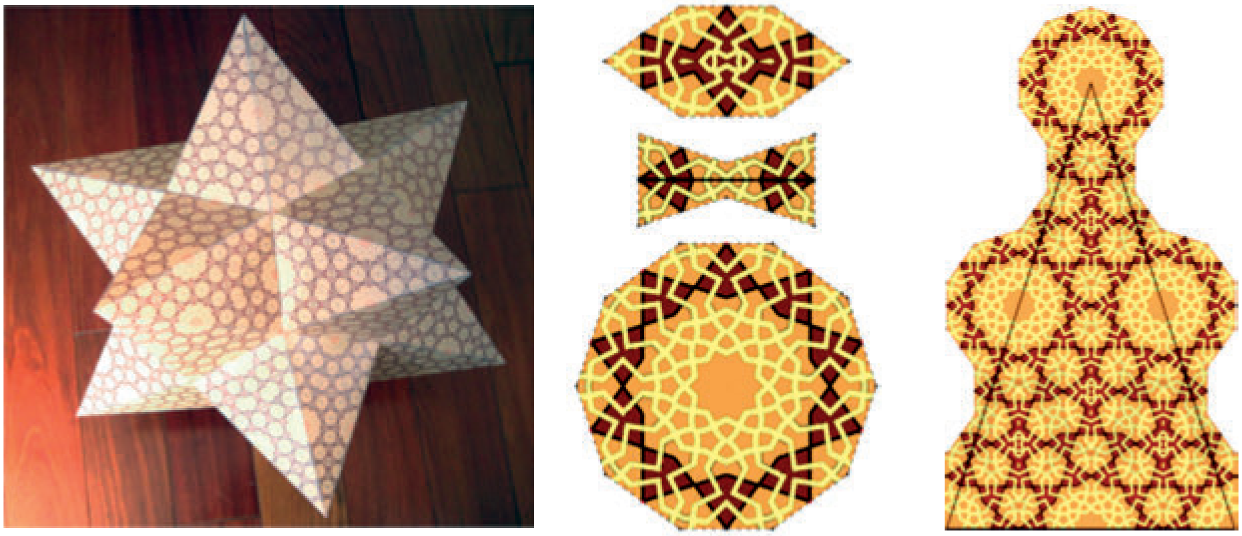


Figure 12: *Dance of Stars I, the hexagon, bowtie, and the decagram, and the golden triangle*

Dance of Stars I in Figure 12 is one of the four Kepler-Poinsot solids, the *small stellated dodecahedron* that the author sculpted that has been ornamented by the *sâzeh* module tiles. The girih tiles were used to create an artistic tessellation for adorning the surface area of the golden triangle. Similar to panel 28 of the Topkapi scroll in Figure 13, the dashed outlines of the girih tiles were left untouched in the final tessellation. The author also included off-white rectilinear patterns that appear as additional small-brick pattern in the 12^{th} century decagonal Gunbad-i Kabud tomb tower in Maragha, Iran, as is shown in Figure 13. For the hexagon and bowtie girih tiles (middle column of images in Figure 12), these additional patterns posses internal two-fold rotational symmetry. But then this symmetry was followed to create a ten-fold rotational symmetry, in order to cover the surface area of the decagonal tiles as well. It is important to mention that the final tessellation had to conform to three essential rules: (1) Each vertex of the triangles had to be the center of the main motif of the tiling, the decagram; (2) The tessellation should be bilaterally symmetric, (3) The sides of the triangles should be the reflection lines of the motifs located on the edges. Without a thorough mathematical analysis of the pattern, it would be extremely difficult, if not impossible, to create a satisfactory artistic solution.

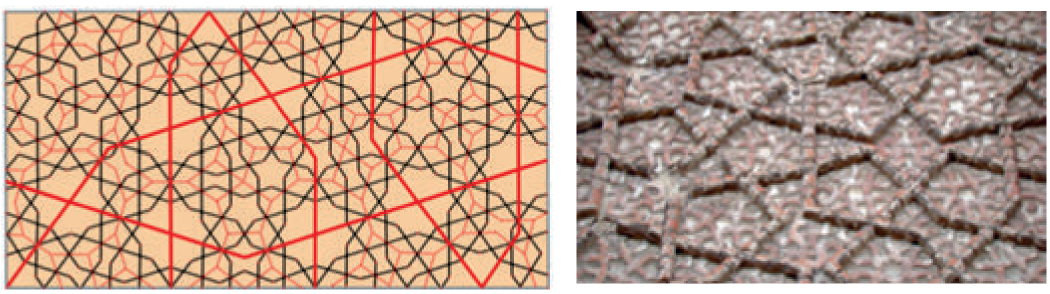


Figure 13: *A rendering of plate 28 in the Topkapi Scroll, and the design on the Gunbad-i Kabud tomb*

Similar to the previous star in Figure 12, *Dance of Stars II* and *III* in Figure 14 are the other two Kepler-Poinsot solids, the *great stellated dodecahedron,* and *great dodecahedron*, which have been decorated by the *sâzeh* module tiles.

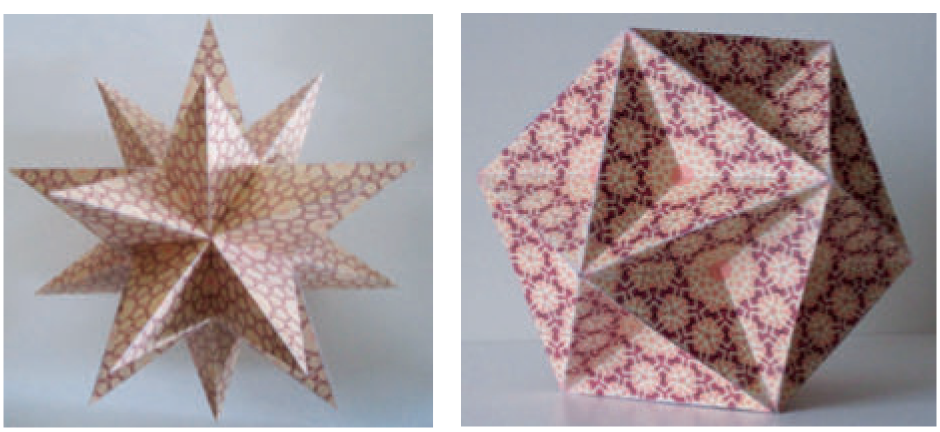


Figure 14: *Dance of Stars II-III, ornamented Great Stellated Dodecahedron* and *Great Dodecahedron*

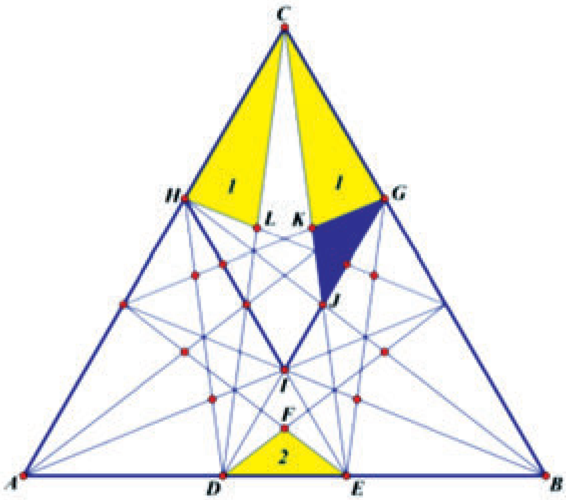


Figure 15: *The stellation pattern of the icosahedron*

The triangles that constitute the *great icosahedron* are different from the previous triangles, which were parts of the regular pentagon. So it is possible that we cannot adorn their surfaces using the *sâzeh* module tiles. To construct the two triangles, we need to start with an equilateral triangle and divide the edges at the golden ratio points to create six new vertices on the edges. We then connect them to make "the stellation pattern of the icosahedron" as illustrated in Figure 15 [11]. The two triangles 1 and 2 are the desired triangles (or 1 and *D JGK,* which is congruent to 2). It remains an open question whether we can have a sâzeh module tiling solution or if such a solution is impossible.

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Activity 1: A Workshop in Constructing a Dodecahedron Using the Mirza Akbar Pentagon

To construct the dodecahedron we need 12 ornamented pentagons (Figure 1). If you would like to make the decorating pattern yourself, follow the construction steps of Figure 2. We also can attach three pentagons to each other to make the cutting and pasting process easier. We then need 4 copies of the attached triangles (Figure 3).

For gluing together the pieces, use transparent adhesive tape.

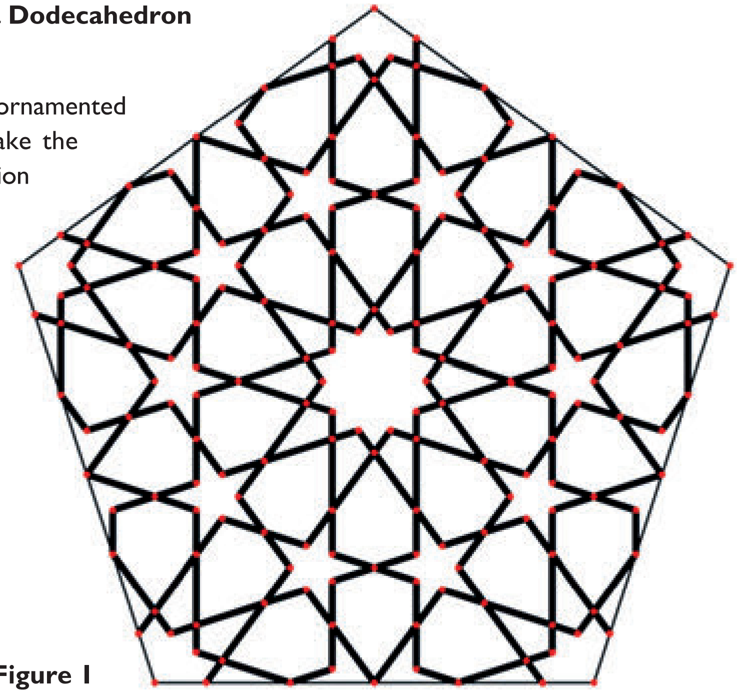


Figure 1

Figure 2

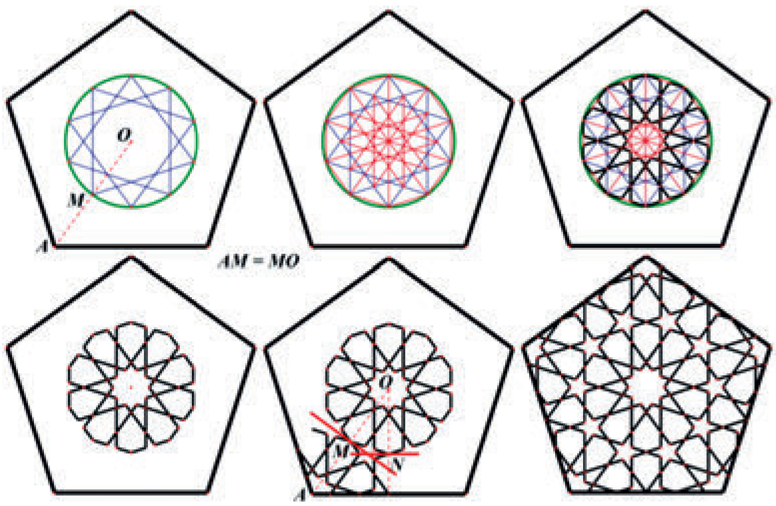
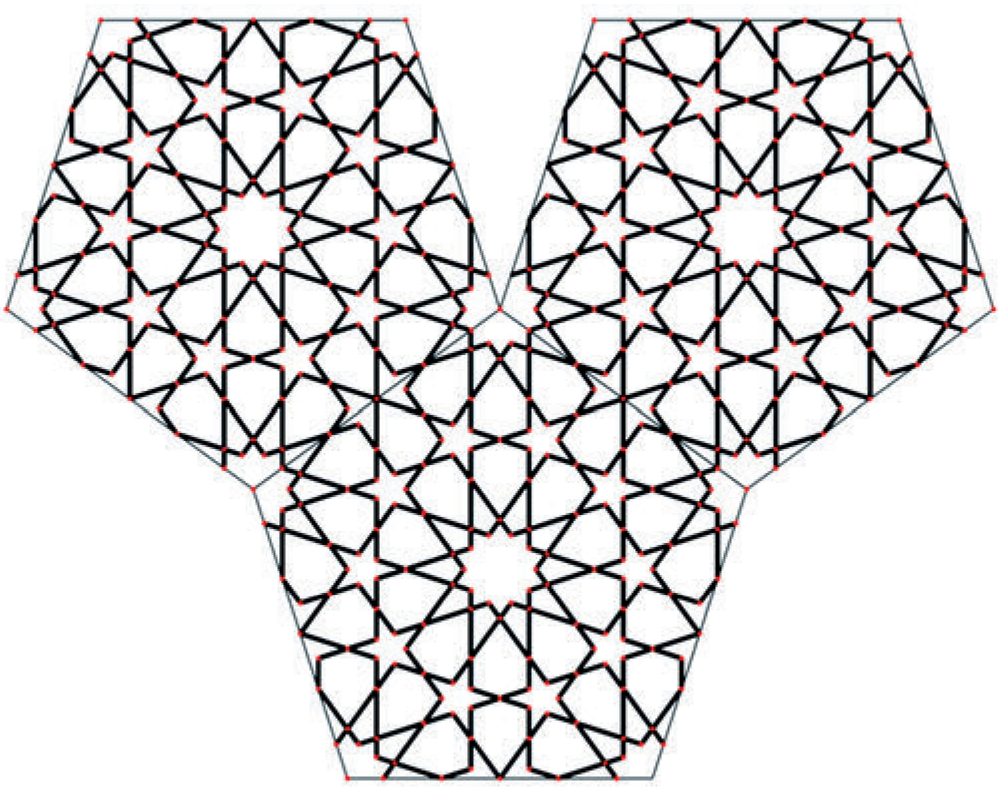


Figure 3



Activity 2: A Workshop in Constructing the Ornamented Kepler-Poinsot Polyhedra

During this activity the workshop participants construct three of the four Kepler-Poinsot polyhedra. These solids are the *small stellated dodecahedron* (Figure 1), the *great stellated dodecahedron* (Figure 2), and the *great dodecahedron* (Figure 7). For this, using *girih* tiles, we construct the shape in Figure 3. Then from Figure 3 we find the ornamented *Golden Triangle* in Figure 4. We need 60 copies of the triangle in Figure 4 for building the *small stellated dodecahedron* (Figure 1) and the same number of copies for constructing the *great stellated dodecahedron* (Figure 2). It is also possible that we use Figures 5 and 6 instead. In this case all the required triangles for a pyramid have already been attached. We then need 12 copies of the polygon in Figure 5 to build Figure 1 and 20 copies of the polygon in Figure 6 to make Figure 2.

For gluing together the pieces, use transparent adhesive tape.

Figure 1: *Small stellated dodecahedron*

Figure 2: *Great stellated dodecahedron*

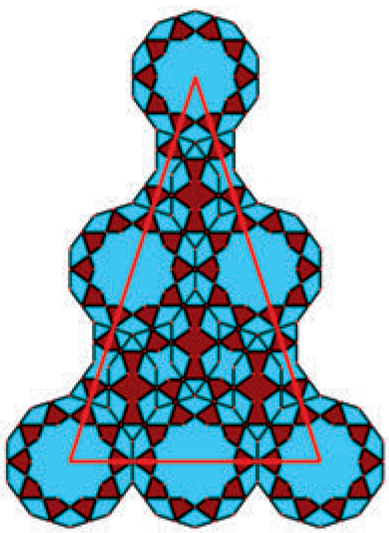


Figure 3: *The girih tiles pattern for the Golden Triangle*

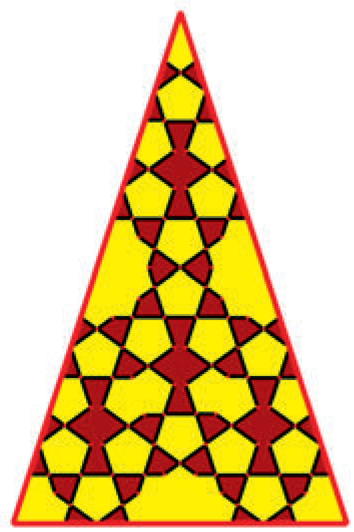


Figure 4: *The ornamented Golden Triangle*

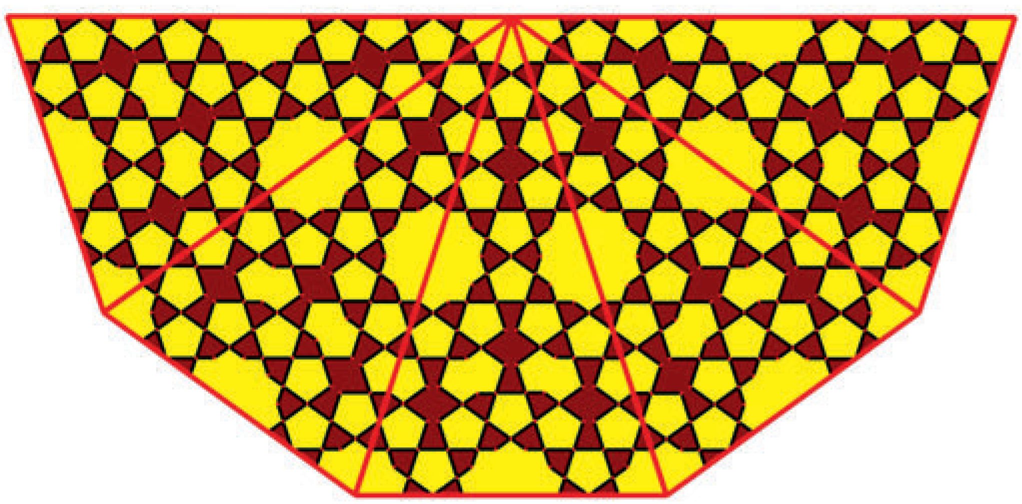


Figure 5: *Five connected Golden Triangles to make one pyramid of one side in Figure 1.*

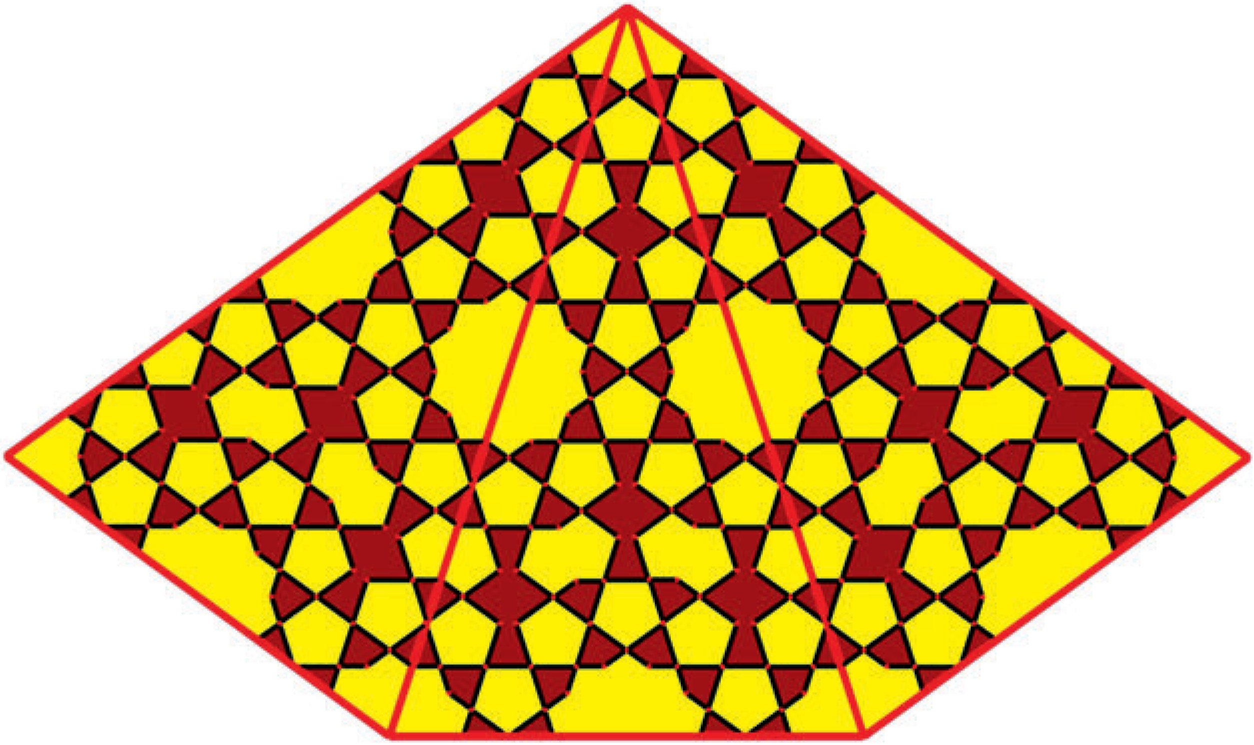


Figure 6: *Three connected Golden Triangles to make one pyramid of one side in Figure 2.*

To construct the *great dodecahedron* (Figure 7) we need 60 copies of the Obtuse Golden Triangle in Figure 9. We also can construct this polyhedron using 20 copies of the polygon in Figure 10. Please note that in this activity the ornamented side of each triangle should be inside of the constructed pyramid.

For gluing together the pieces, use transparent adhesive tape.

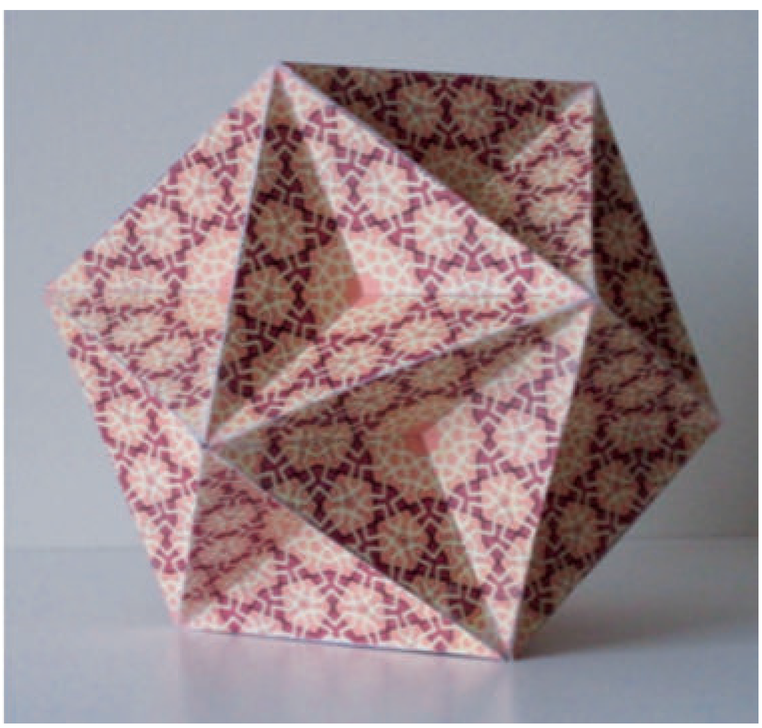


Figure 7: *great dodecahedron*

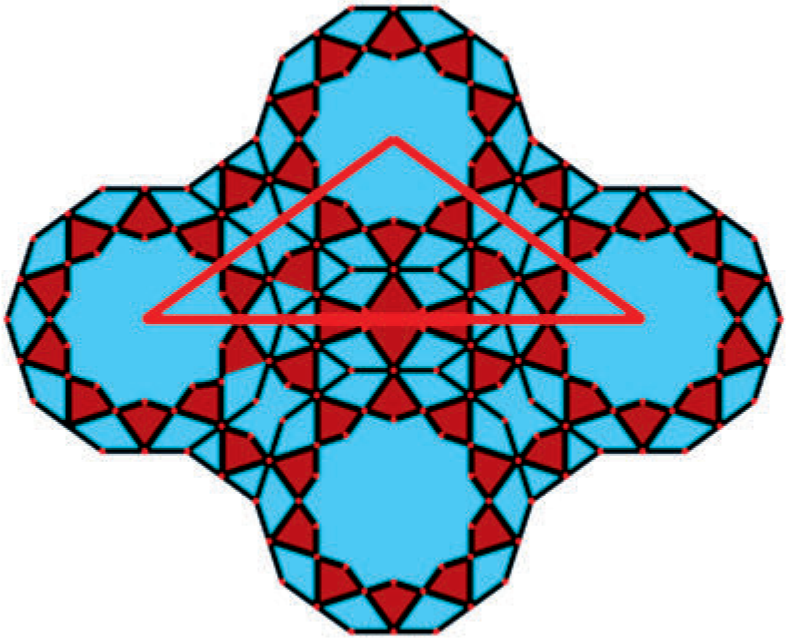


Figure 8: The pattern for the Obtuse Golden Triangle

Figure 9: The Obtuse Golden Triangle

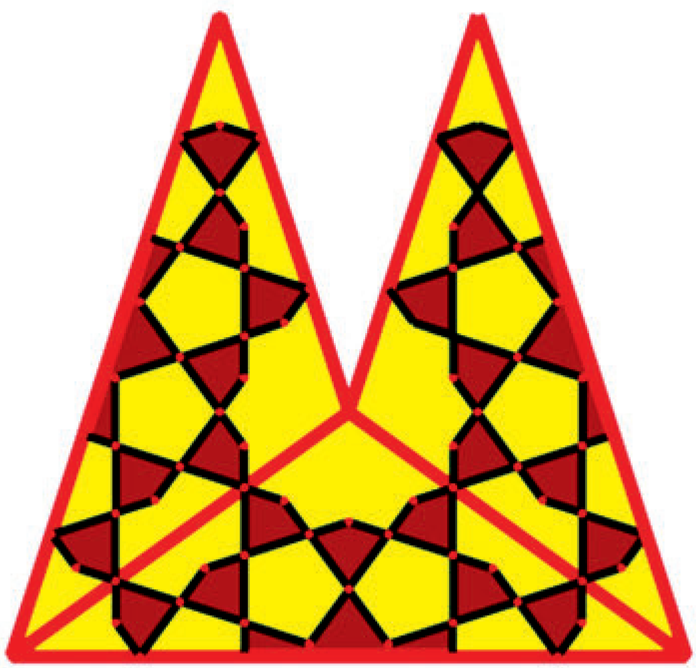


Figure 10: *Three connected Obtuse Golden Triangles to make one pyramid of one side in Figure 7.*

It is important to mention that for Figure 1, we should connect 3 copies of the pyramids around each vertex. For Figure 2, we need 5 copies of the pyramids around each vertex. For Figure 7, we also need 5 copies of the pyramids around a vertex. We also may use the following two ornamented triangles that include extra patterns (Figure 11, 12).

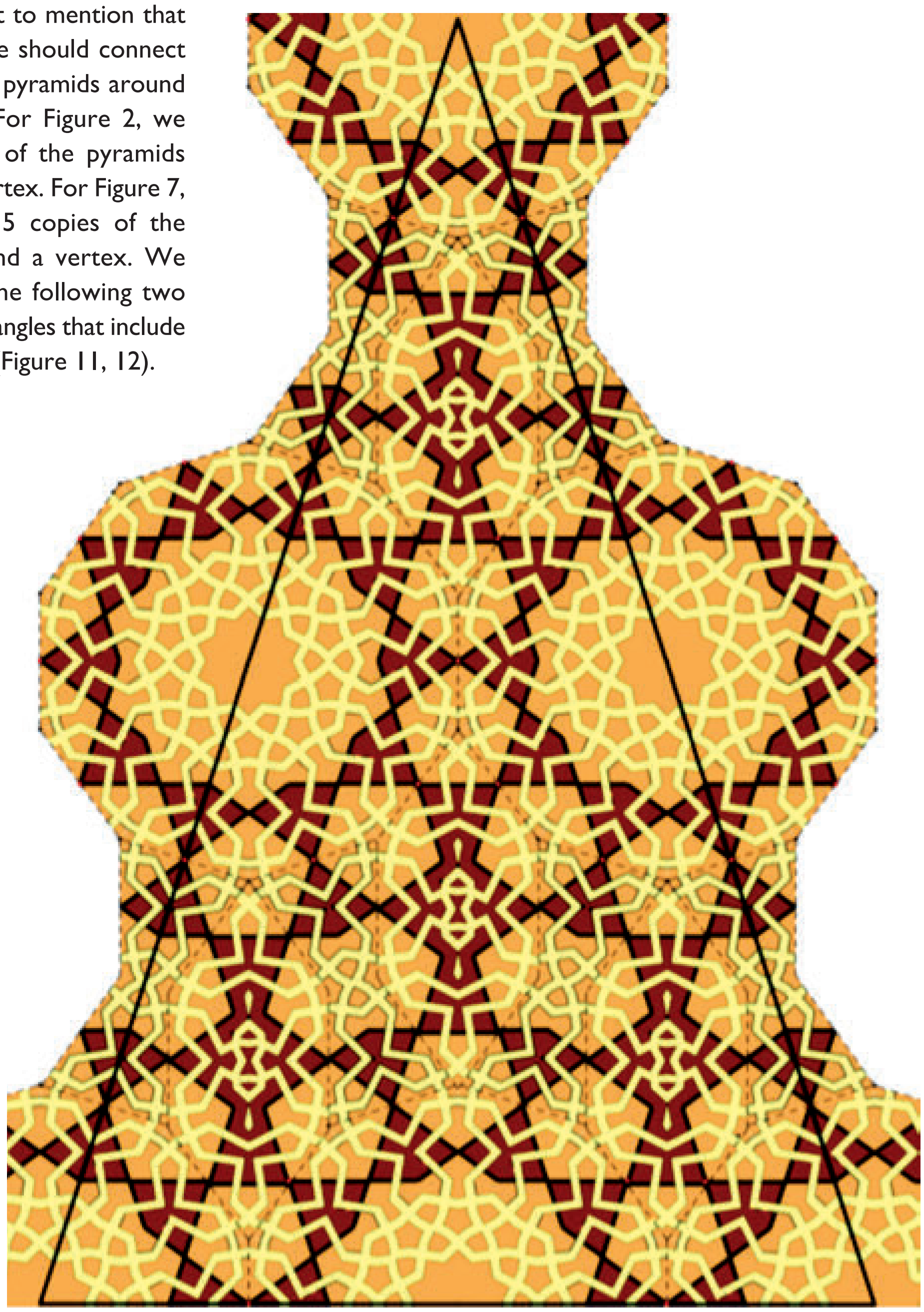


Figure 11: The Ornamented Golden Triangle with extra lines

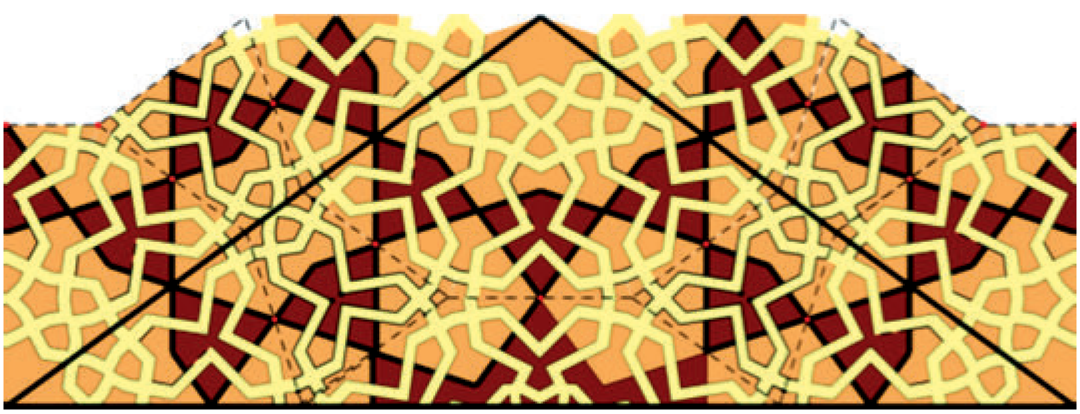


Figure 12: The Ornamented Obtuse Golden Triangle with extra lines