



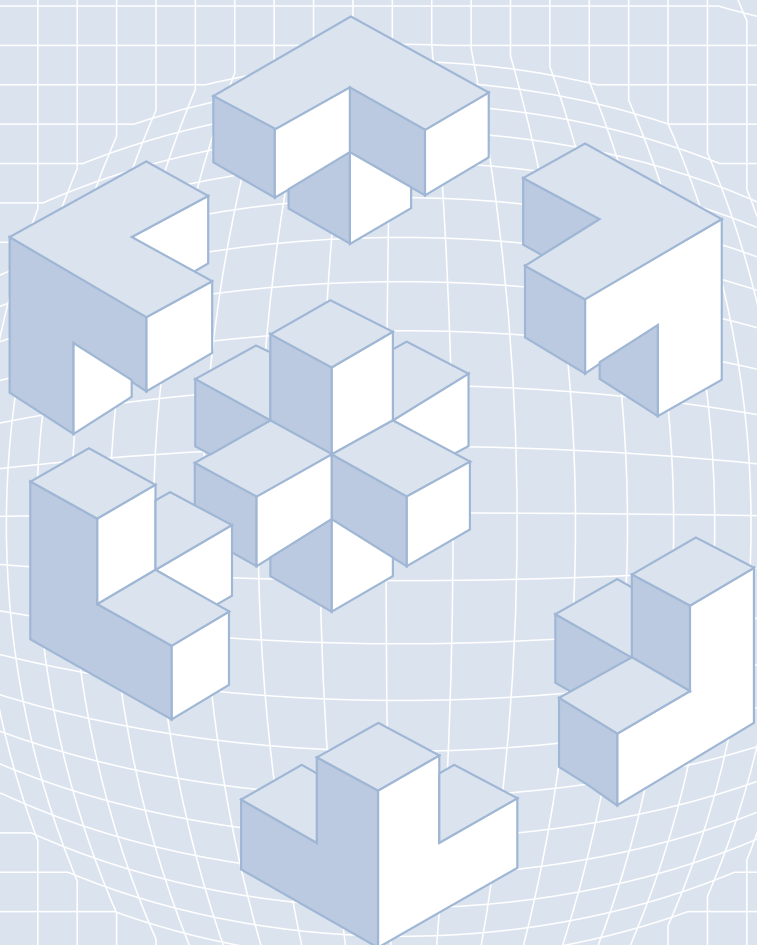
VISUALITY &  
MATHEMATICS

EXPERIENTIAL EDUCATION  
OF MATHEMATICS THROUGH  
VISUAL ARTS, SCIENCES  
AND PLAYFUL ACTIVITIES

TAMÁS F. FARKAS

# IMPOSSIBLE OBJECTS, PARADOX FIGURES

TRICKY STRUCTURES, PLAYFUL  
PERSPECTIVES





The artist **TAMÁS F. FARKAS**'s creative career is as outstanding and unconventional as you can get in contemporary art. According to Dr. György Darvas, a researcher of Symmetry Studies: "Farkas's work defies categorization. If we want to label it – although this is never going to be able to capture its uniqueness – then perhaps we could proceed from M. C. Escher's activity, that is, geometry is always at the root of his works. However, while Escher's geometry is always connected in some way to figurative representations (strange buildings, animal and human figures), Farkas's work eschews any link to the 'real'. His art is built on clear-cut, geometric figures. He does not want to remind us of real objects, living creatures; the substance of his pictures is about governing

the space or the structure of the space when either real shapes or a projection of higher dimensional figures in the plane are concerned or even while venturing upon the unreal. It is a new »suprematist« art at the end of the 20<sup>th</sup> century but Farkas's work bears no resemblance to that of his suprematist predecessors. He cannot be considered as their disciple as his focus on basic geometric forms is built on his own research work. His pictures are the culmination of years of extensive study in geometry. Therefore Tamás F. Farkas's career is considerable, not only from the point of view of art but also science; he looks at geometry not with the eye of a scientist nor that of an artist but both, thus achieving results that can be utilised in unique ways, ways that would never have been possible with traditional geometry. So his works can, for example, be used to illustrate phenomena of other sciences based on mathematics (such as crystallography, quantum physics etc.), allowing us to better understand abstract-rational results incomprehensible by sensory organs. Farkas has a unique gift but due to its nature – however much his results are appreciated – science cannot accept him as a scientist". Whilst he represents a scientific-minded creator among artists, he is an artist among scientists.

**Tamás F. Farkas's website:** [www.farkas-tamas.hu](http://www.farkas-tamas.hu)

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## 1 GET ACQUAINTED WITH IMPOSSIBLE OBJECTS!

An impossible object is a kind of optical illusion. A two-dimensional image visually perceived as a planar projection of a three-dimensional object, however after observing the image carefully it turns out that the structure represented in two dimensions has such geometric features that cannot be implemented in three dimensions, this is called an impossible object.

There are several noted examples of impossible objects from the fine arts, but the phenomenon is equally fascinating and challenging to psychologists and mathematicians too.

Impossible objects were scientifically described for the first time by psychiatrist, Lionel Penrose and his son, the later world famous mathematical physicist Roger Penrose, in their paper: *Impossible Objects: A Special Type of Visual Illusion* (1956). The paper was illustrated by the impossible triangle (figure 1) and the impossible steps (figure 2) and these were also used by both the Swedish painter Oscar Reutersvärd and the Dutch artist M. C. Escher in their works.

Figure 1: Penrose's impossible triangle

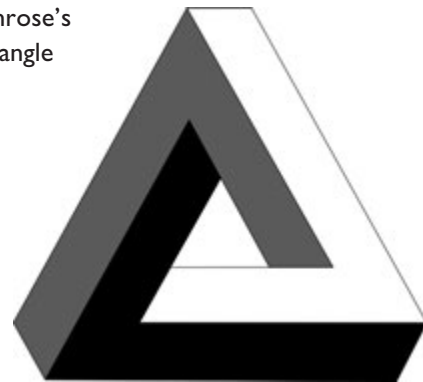
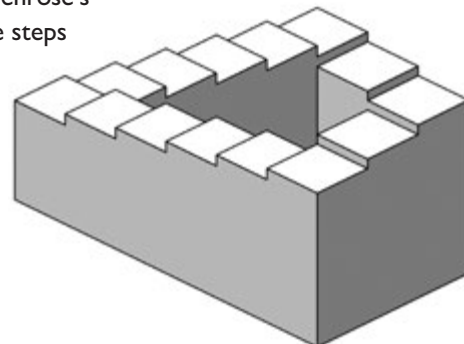


Figure 2: Penrose's impossible steps



In the case of impossible objects a specific correspondence develops between the two and three dimensional space. Therefore studying or drawing these solid objects can play an important role in visual art training as well as in mathematics education. Studying impossible objects not only helps in thinking creatively but it also improves depth perception. Furthermore, getting acquainted with impossible objects can open the way to understanding higher (upper than 3) dimensional spaces and the conceivable structures within them.

*Exercise: study Tamás F. Farkas's paradox artworks as structures in space!*

*Why are these structures paradoxical? Study and think about their planar and spatial representation!*

*Which parts of the structures are possible and which are impossible to create in the space?*



Figure 3: Tamás F. Farkas's artworks as two-dimensional visual representations of paradox three-dimensional structures



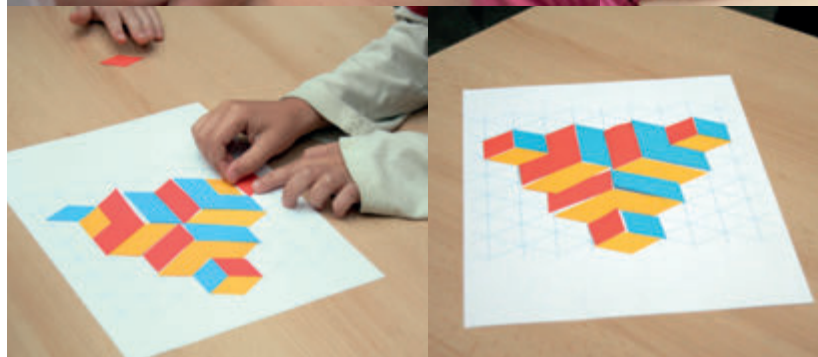
## 2. CREATING IMPOSSIBLE OBJECTS: IS IT POSSIBLE? TRY IT!

The first step of the activity is to observe thoroughly the artwork of the chosen template. With students of an appropriate age it is also possible to analyse the given artwork geometrically, too.

Copies of templates A and B belonging to the given artwork are printed according to the number of participating students. The figure on template A is cut into elements along the black line bordering elements with a pair of scissors. Afterwards, the students' task is to recompose the figure on the raster net B belonging to the given form. It may help if the facilitator of the activity draws the participants' attention to the rule that two elements of the same colour cannot border each other. After completing a figure it is worth both giving a verbal description of the creation, defining their specific geometric features and discussing observations obtained during the construction together. The raster net B can also be used by students to draw the figure as well. After becoming familiar with the geometric features of impossible objects, students can try to design impossible objects on the raster net B independently too.

Some information to aid the facilitator of the workshop in discovering geometric features of each figure are as follows:

- **Template 1AB: Pyramid** - the illustrated three-dimensional figure can be divided into three symmetric parts.
- **Template 2AB: Paradox Triangles** – the two paradox triangles are symmetric diagonally.
- **Template 3AB: Paradox Three-dimensional Form** – the three-dimensional elements visualise a three-dimensional form that closes back in itself in the plane.
- **Template 4AB: Relationship of Paradox Forms** – two distinct paradox forms are chained together.
- **Template 5AB: Impossible Object** – elements that seem to be three dimensional are chained together in the planar representation constituting a continuous paradox figure. Three identical elements generate a form that closes back in on itself.
- **Template 6AB: Relationship of Paradox Forms II.** – two diagonally symmetric paradox spatial forms are chained together into one structure.
- **Template 7AB: Paradox Figure** – in the two-dimensional illustration, the illustrated three-dimensional figure appears when viewed from both beneath and from above, thus causing a paradox phenomenon.

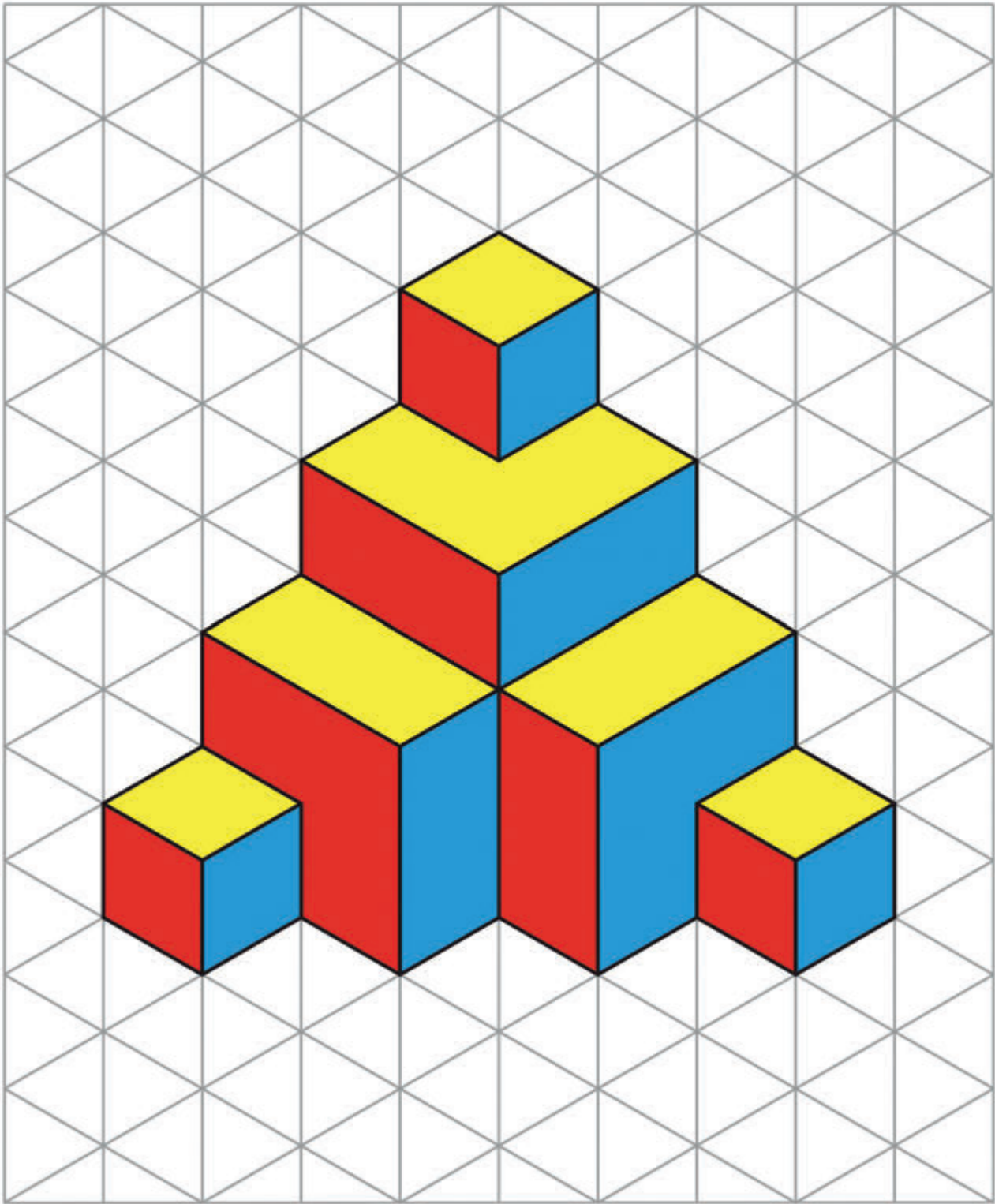


Figures 4-10: Building a pyramid with lower primary school pupils in F. Farkas Tamás's workshop in Experience Workshop – International Movement of Experience-Centred Mathematics Education ([www.experienceworkshop.hu](http://www.experienceworkshop.hu)) event at ANK School in Pécs. Photos: Csaba József Szabó.

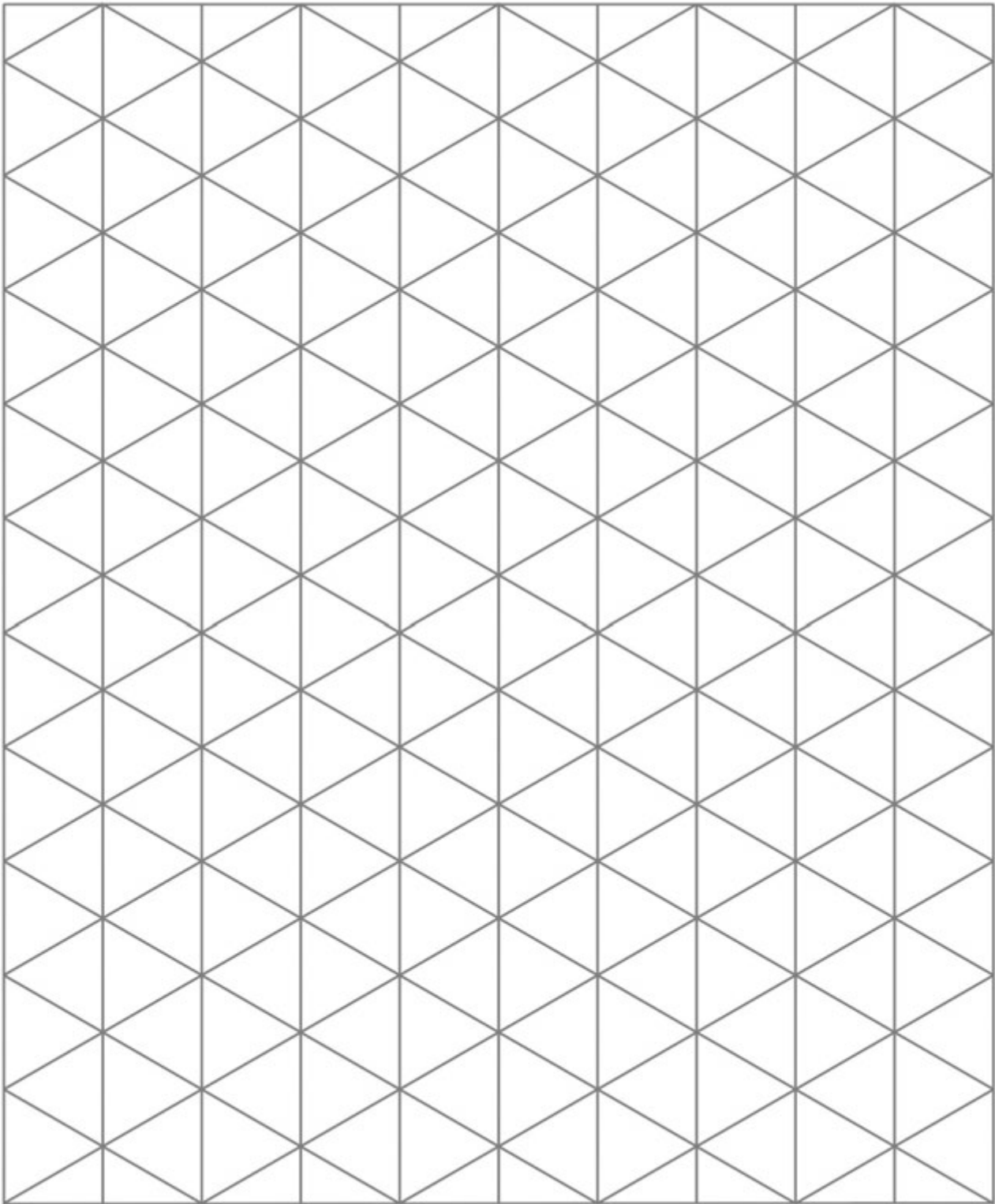




Template 1A: Pyramid

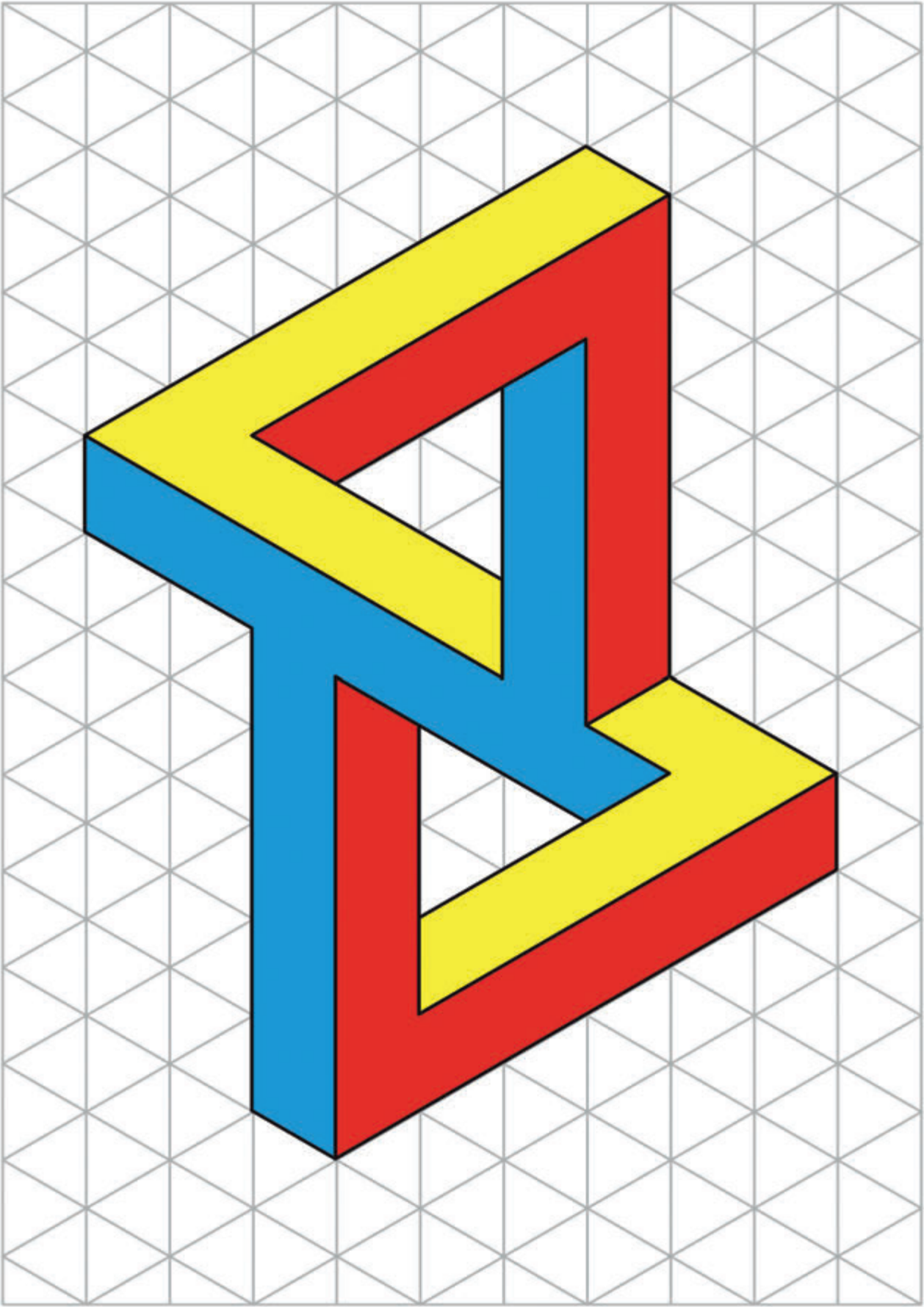


**Template 1B: Pyramid**

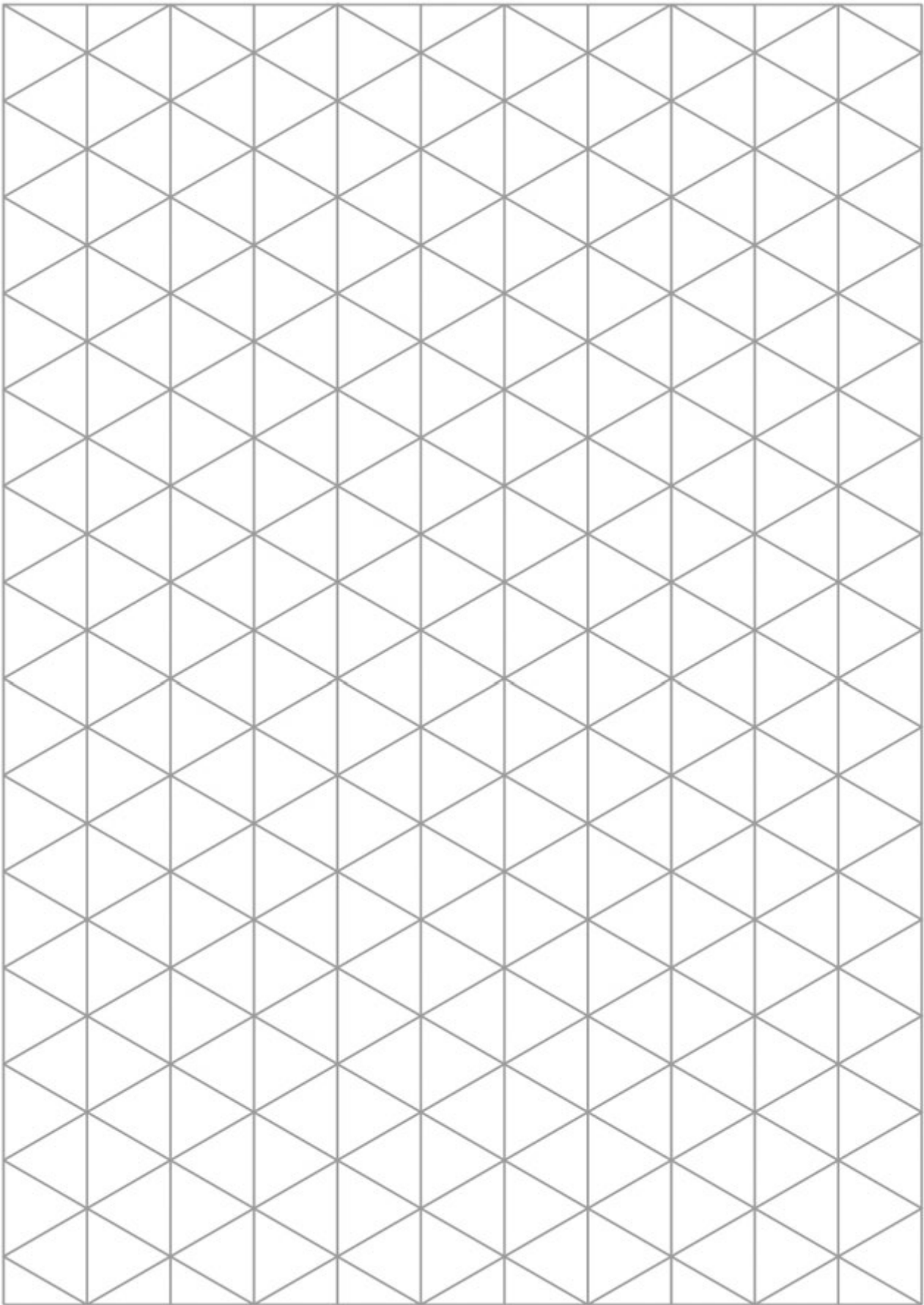




Template 2A: Paradox Triangles

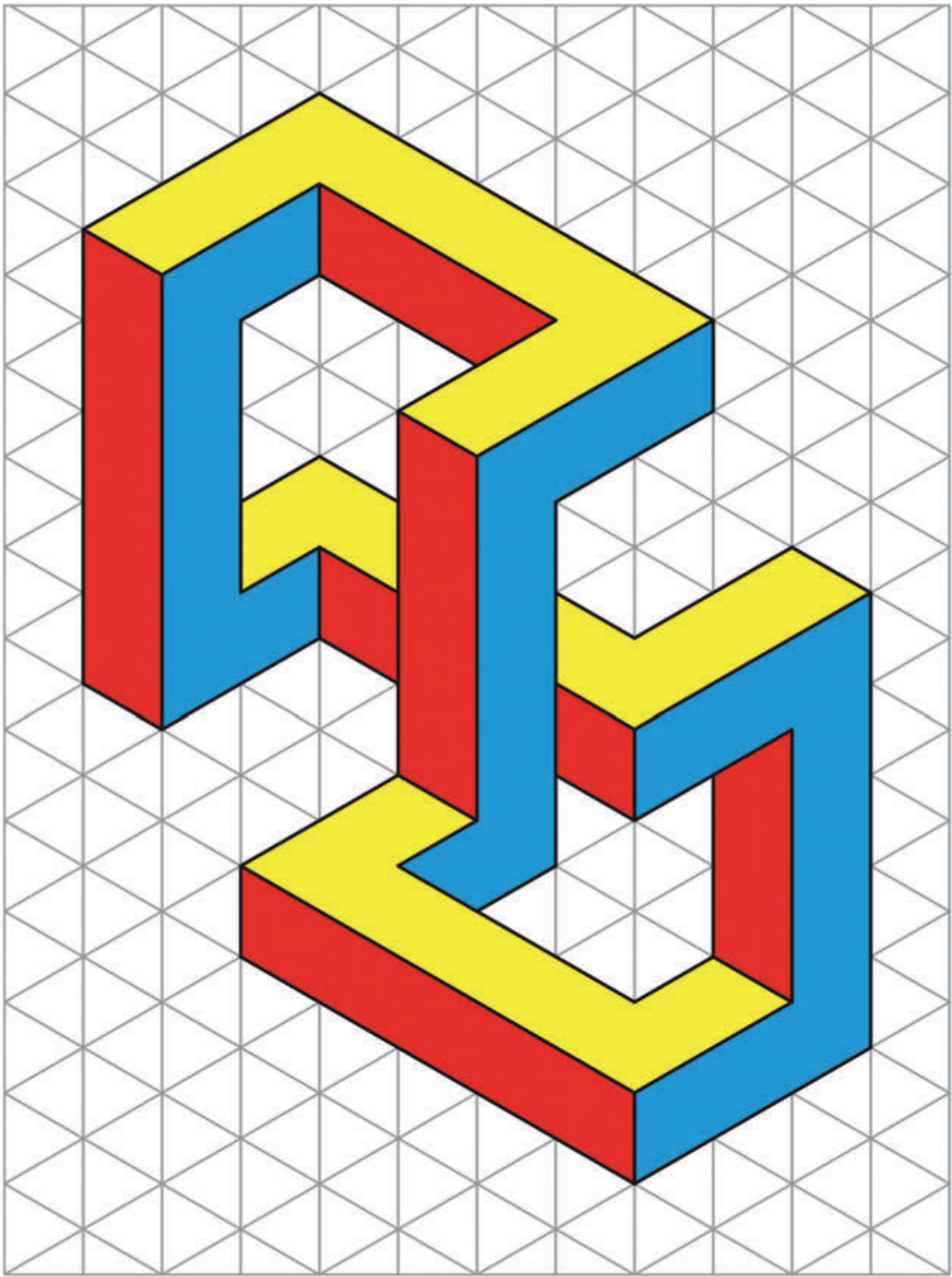


**Template 2B: Paradox Triangles**

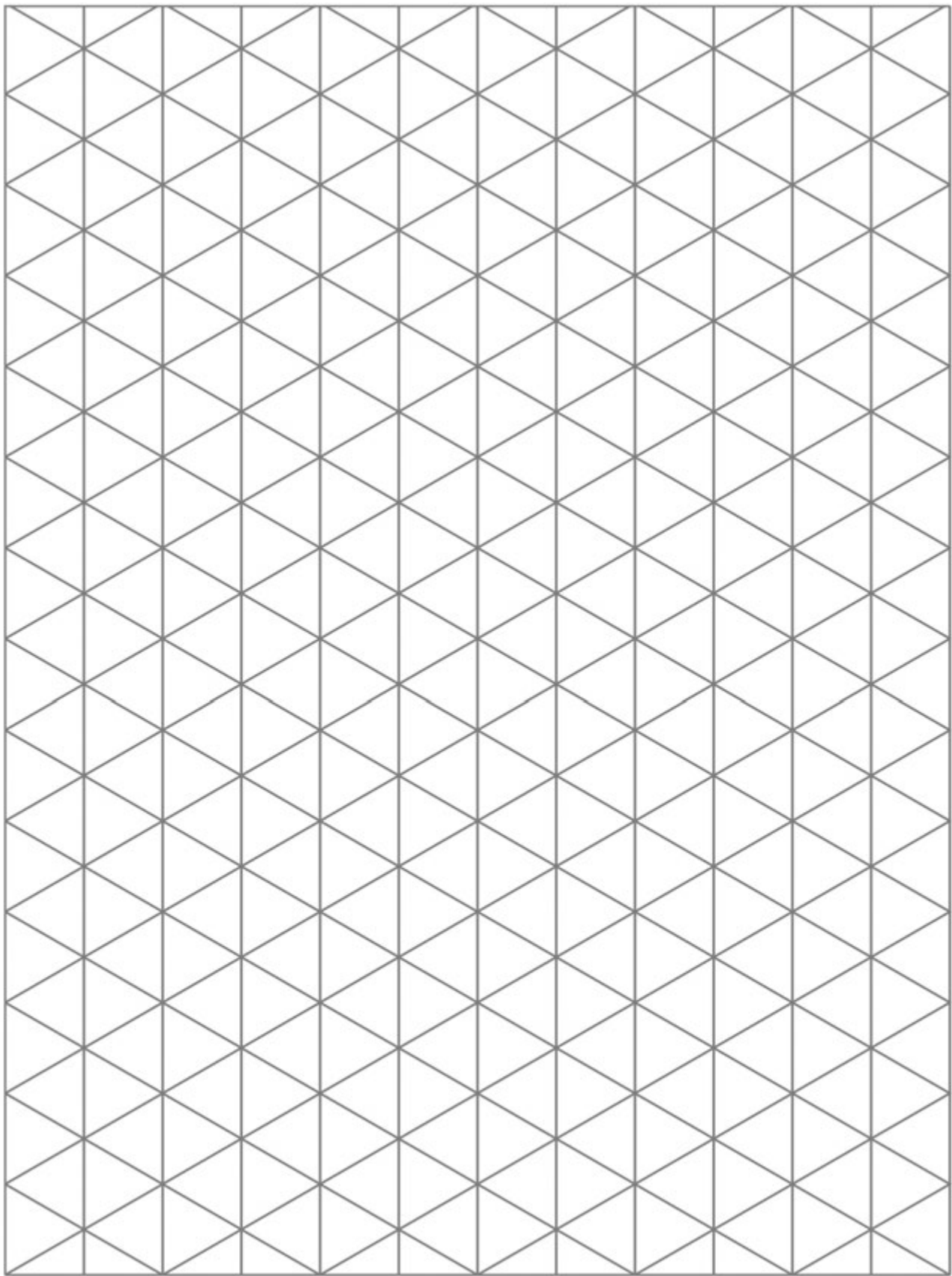




Template 3A: Paradox Three-dimensional Form

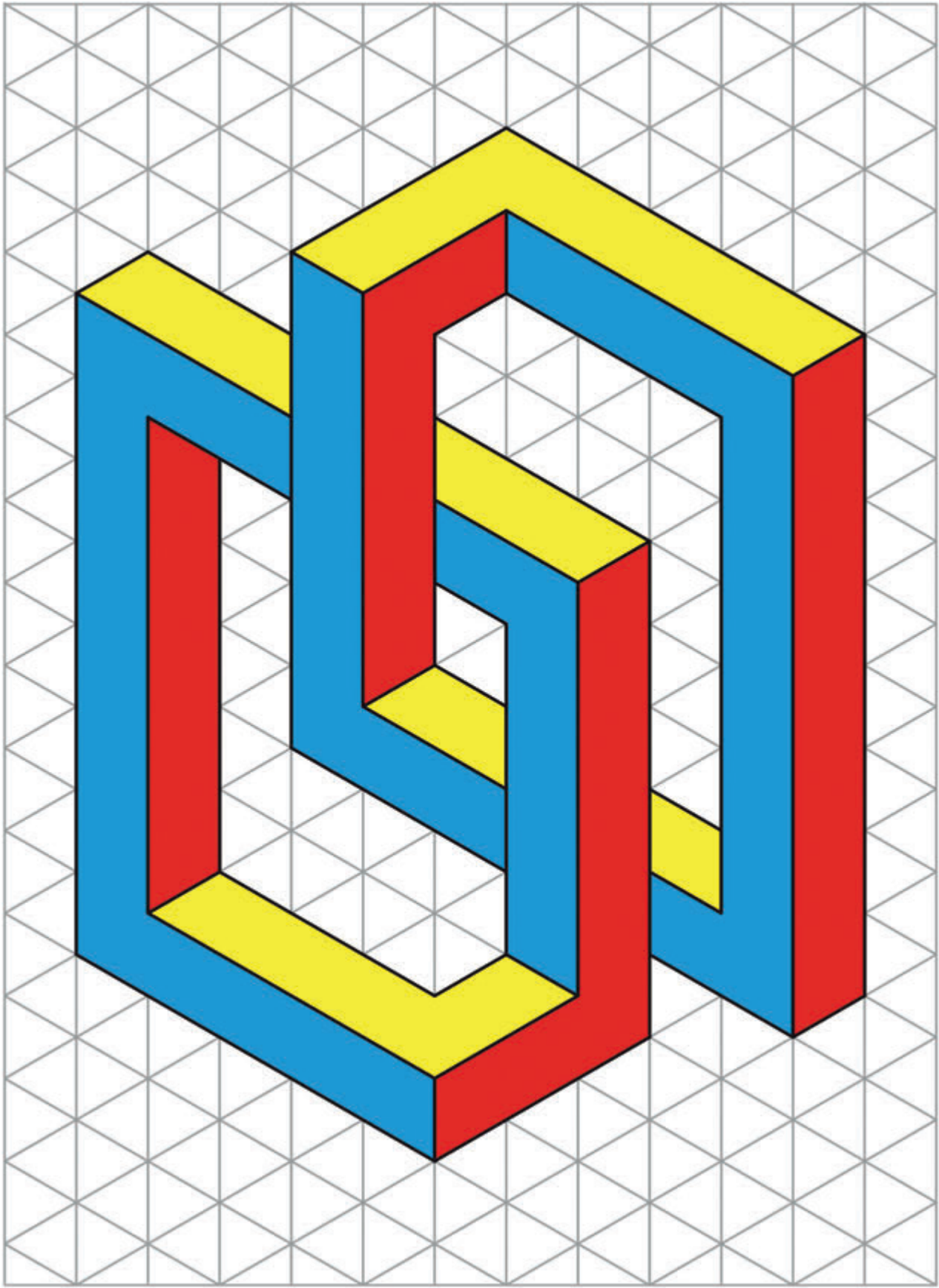


**Template 3B: Paradox Three-dimensional Form**

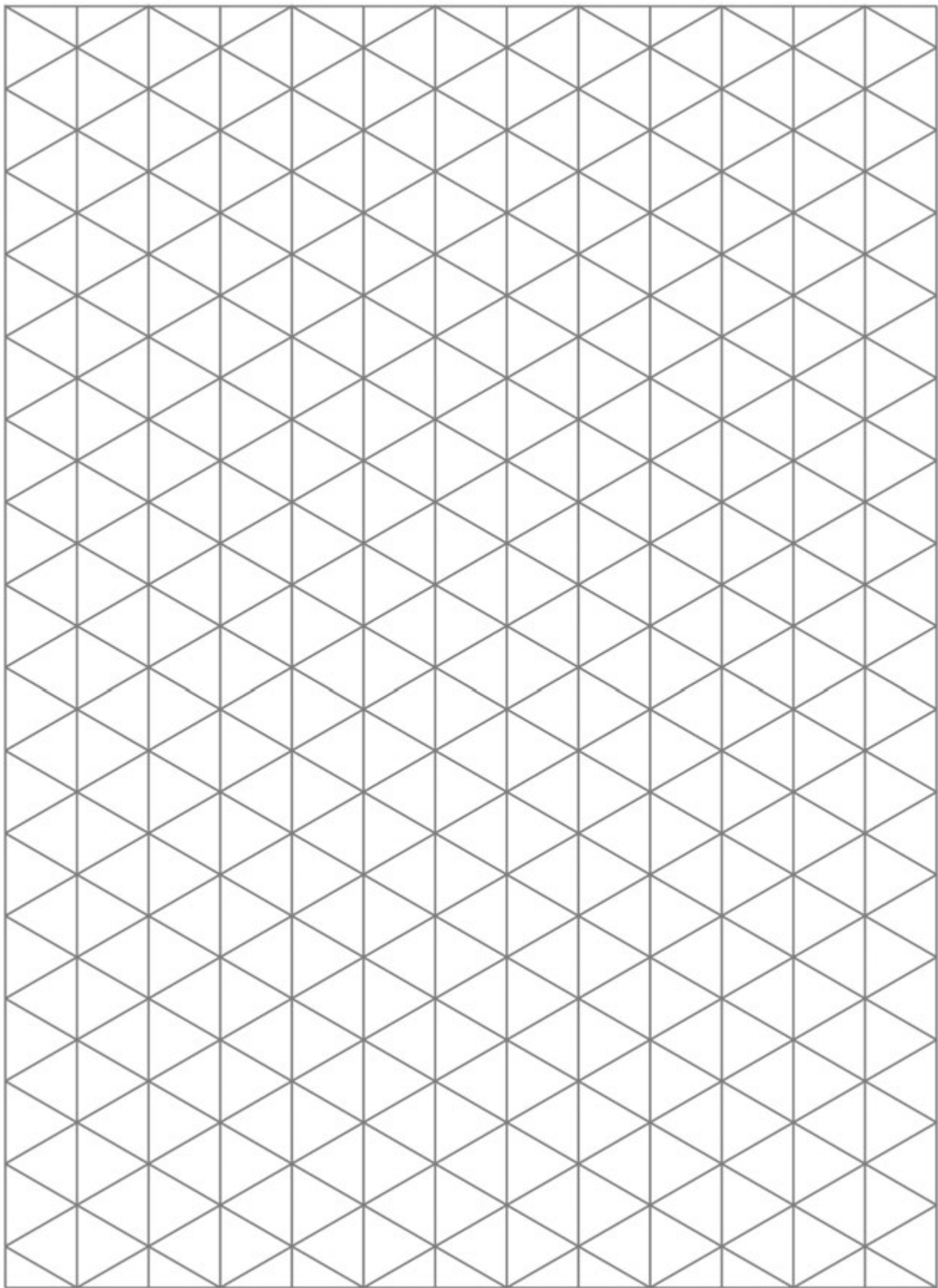




Template 4A: Relationship of Paradox Forms

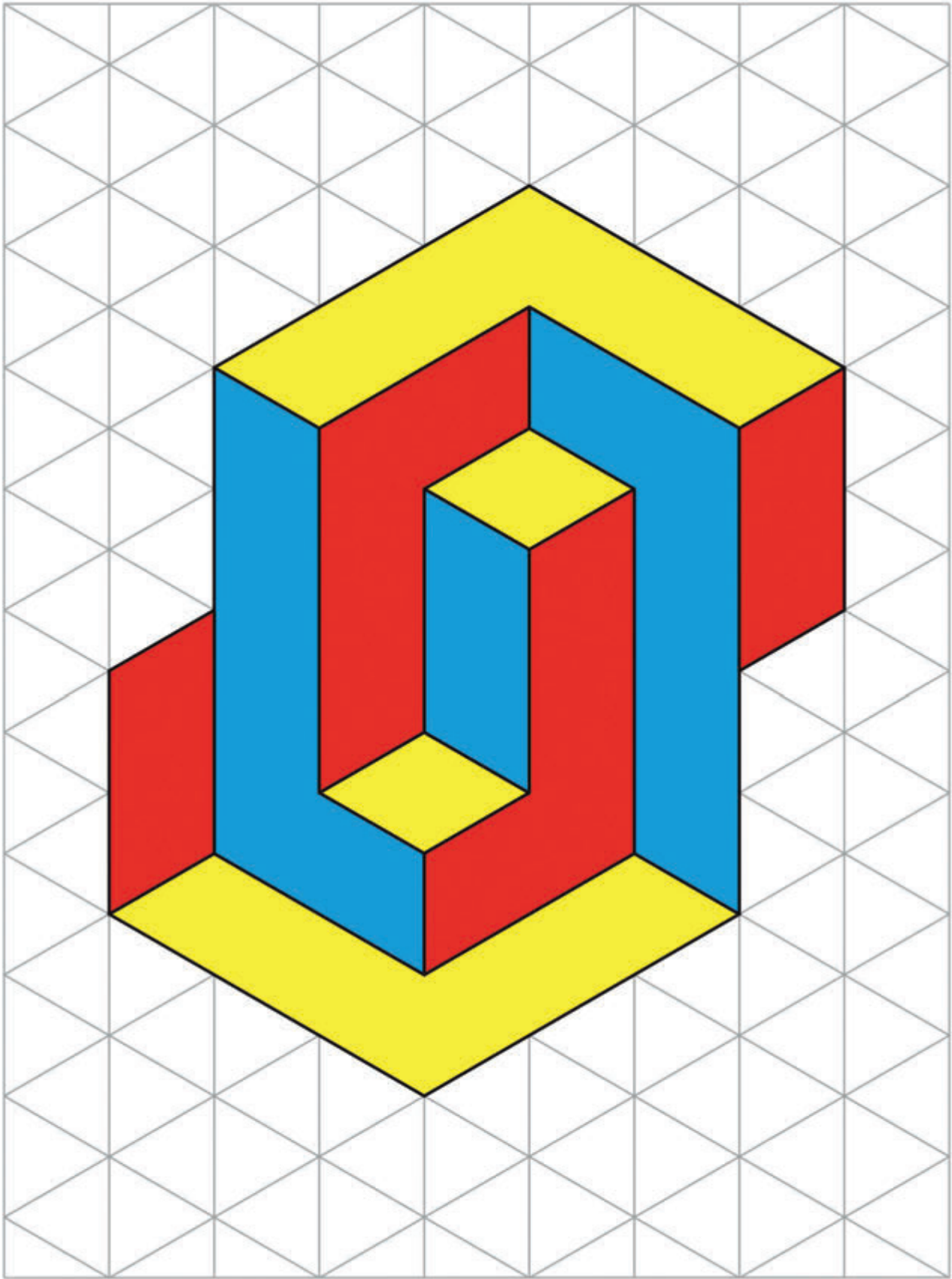


**Template 4B: Relationship of Paradox Forms**

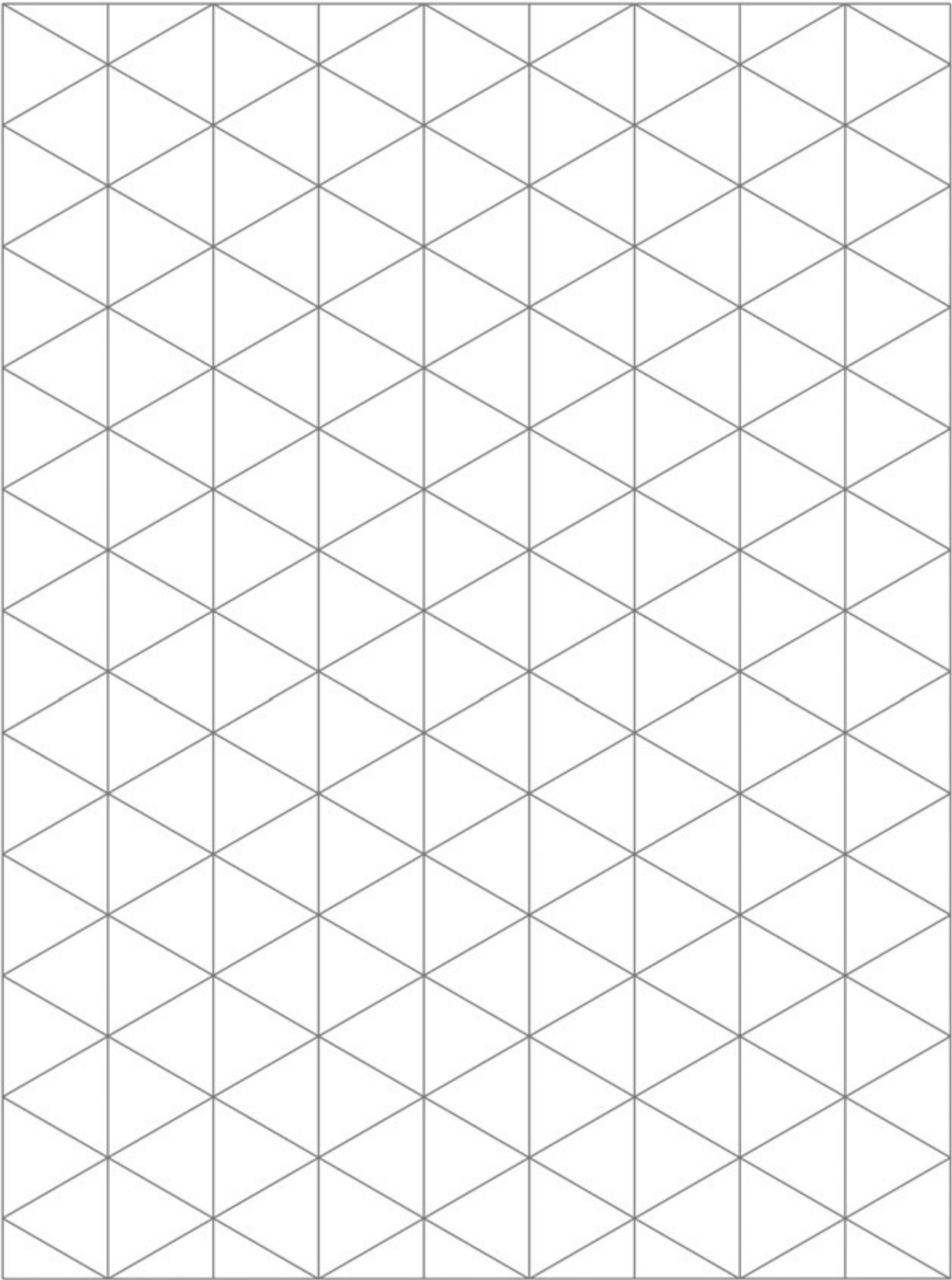




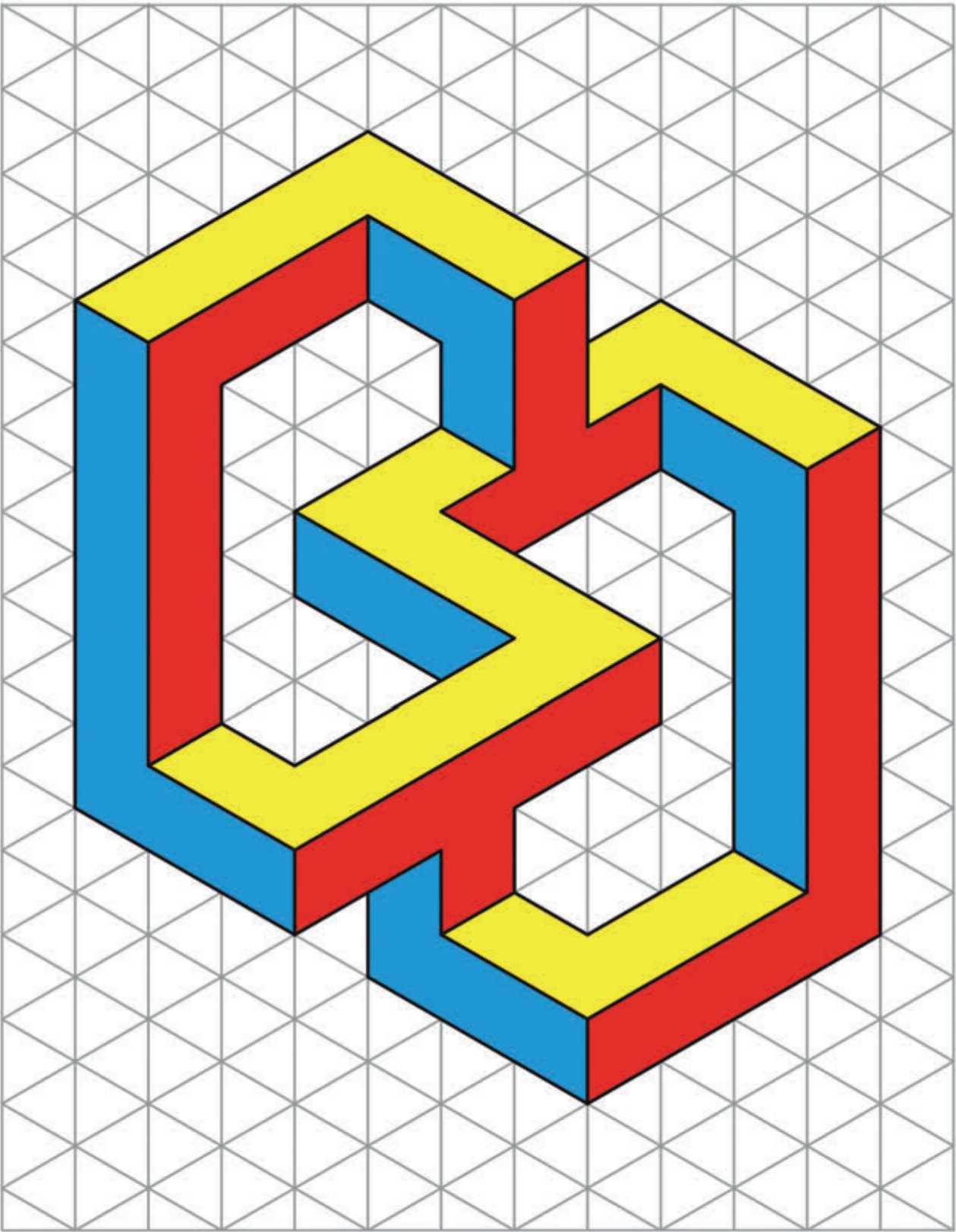
Template 5A: Impossible Object



Template 5B: Impossible Object

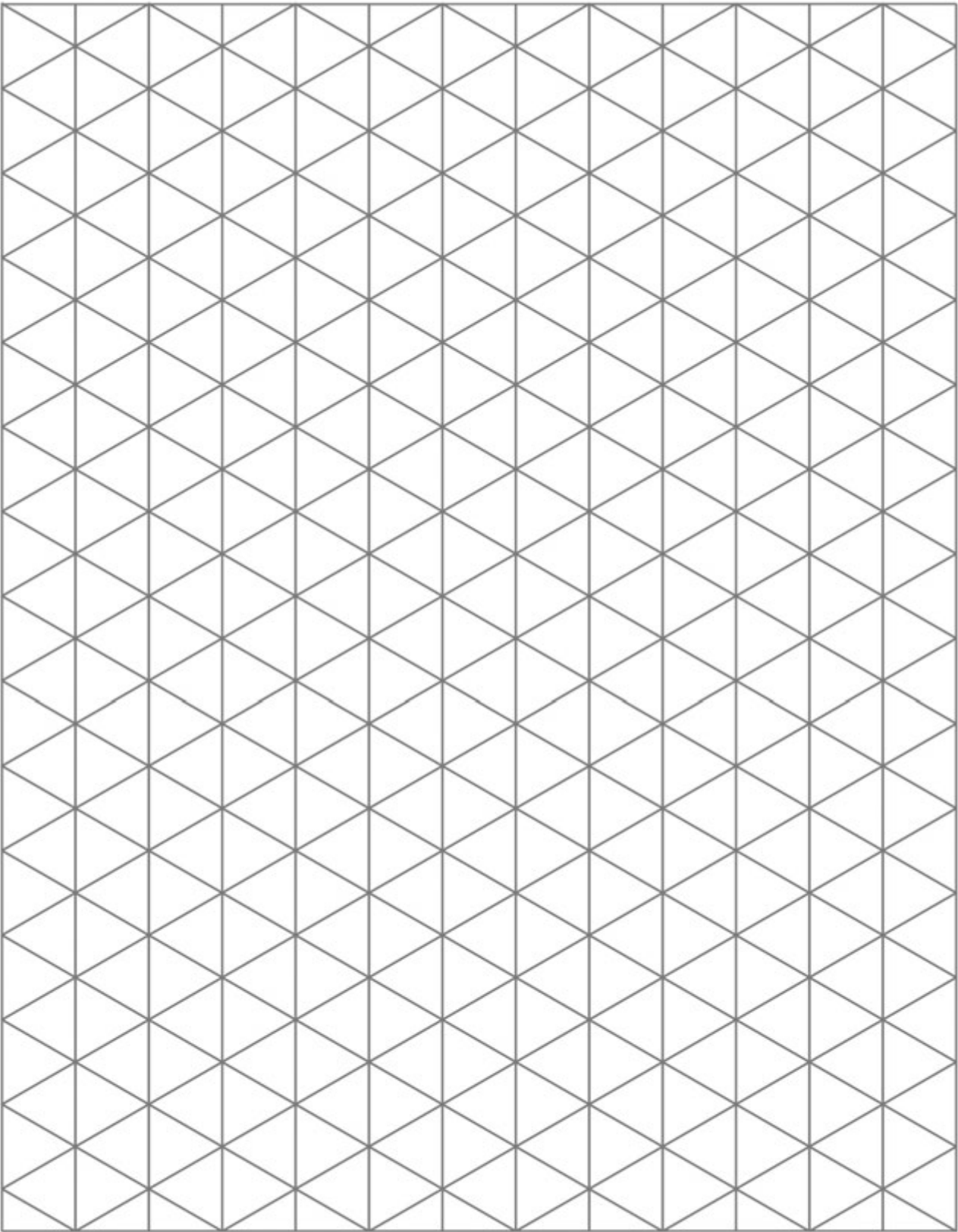


Template 6A: Relationship of Paradox Forms II.

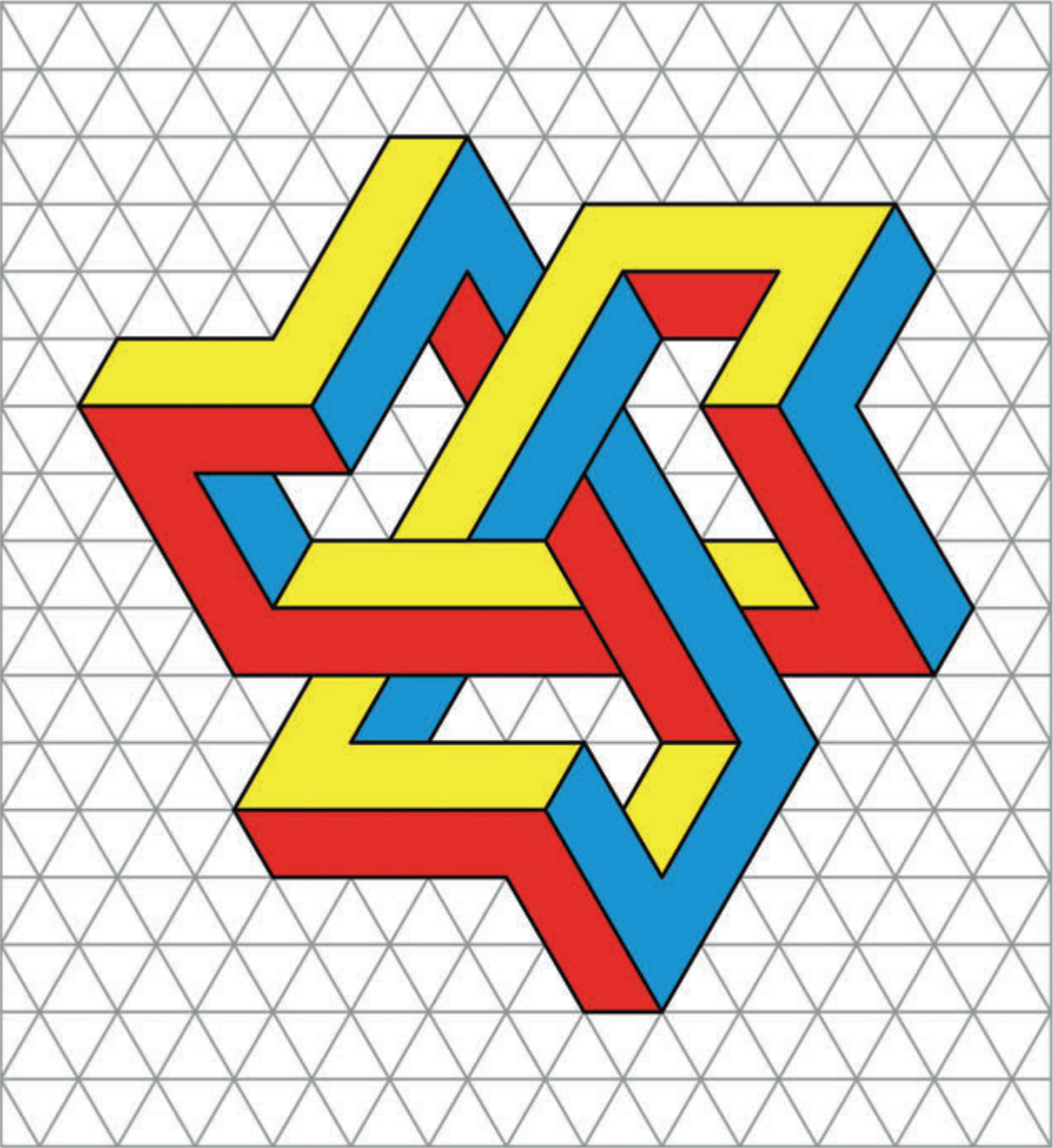




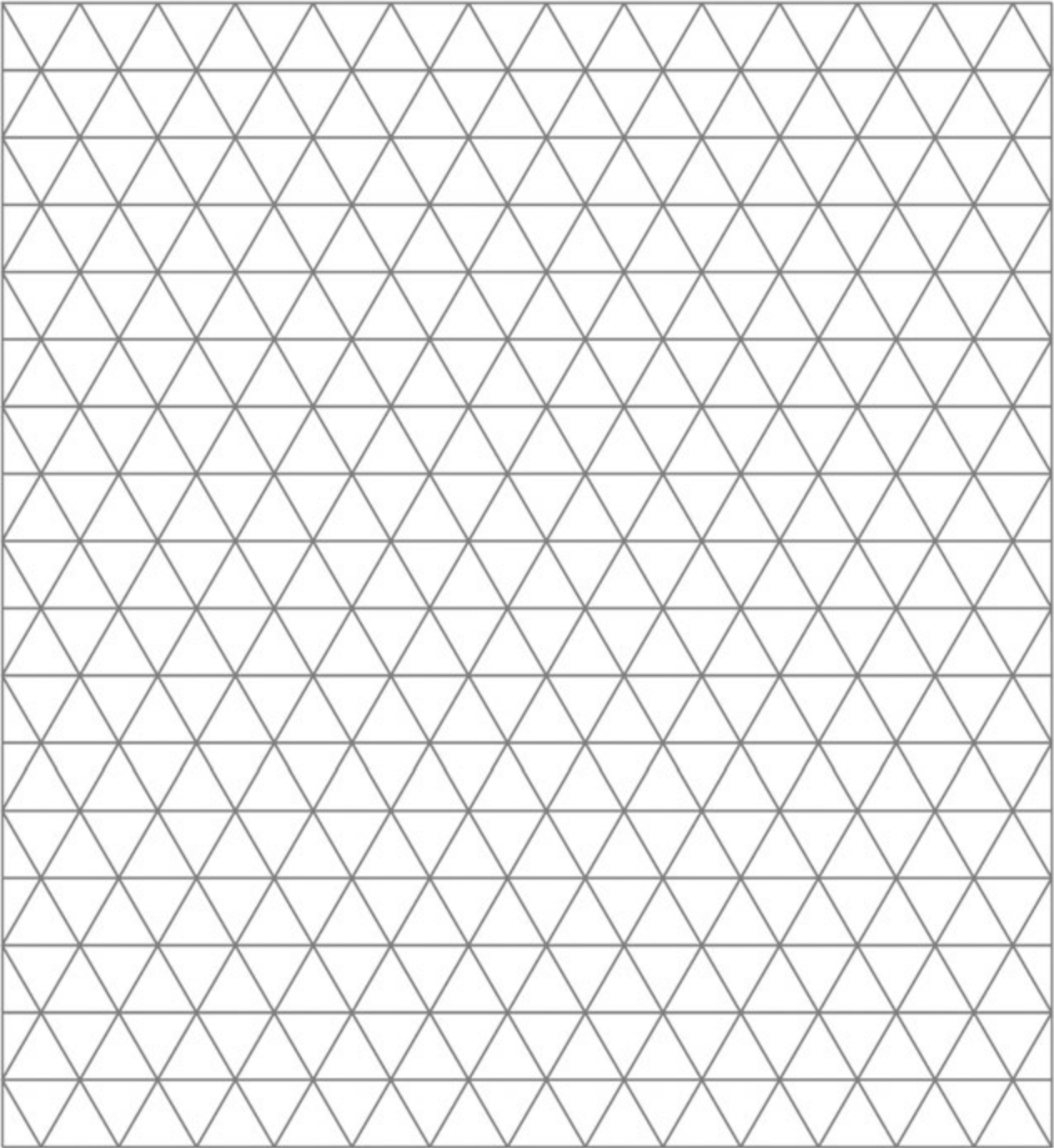
**Template 6B: Relationship of Paradox Forms II.**



Template 7A: Paradox Figure



**Template 7B: Paradox Figure**





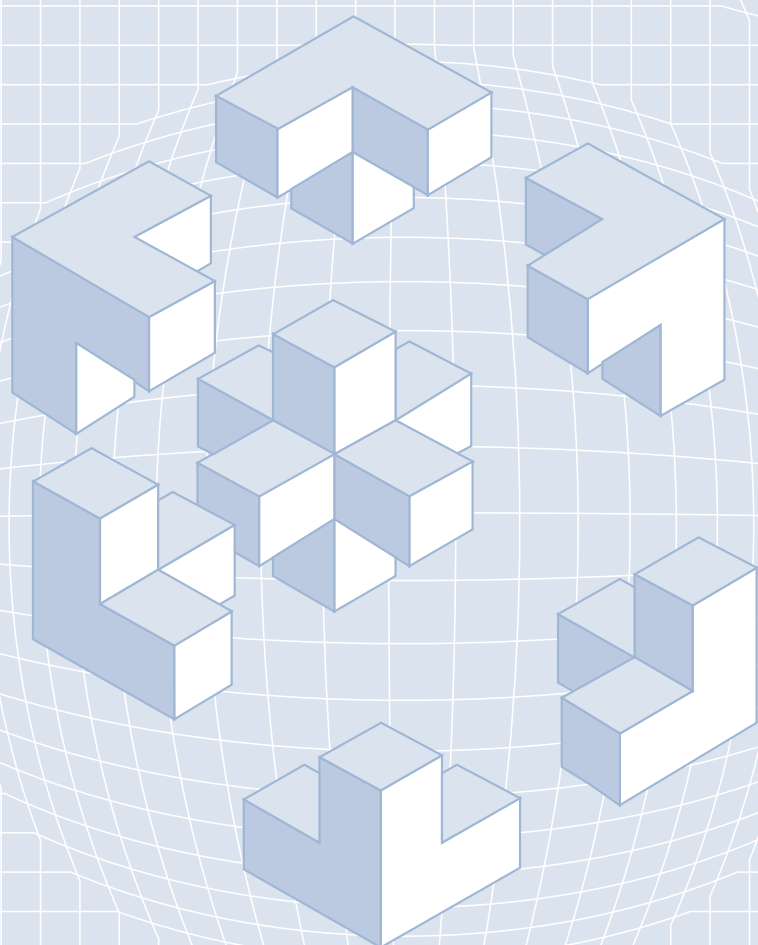


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GEORG GLAESER-LILIAN WIESER

# LET US PLAY NATIVE AMERICAN!



TRICKY STRUCTURES, PLAYFUL  
PERSPECTIVES



**GEORG GLAESER** was born 1955 in Salzburg, Austria. He studied mathematics and received his PhD in geometry, before completing his habilitation in computational geometry at the Technical University of Vienna. Glaeser was a visiting fellow at Princeton University before becoming a tenured professor at the University of Applied Arts Vienna in 1998. Author of more than a dozen books in different fields: Computational Algorithms, Mathematics, Geometry, Photography, Evolution Biology.

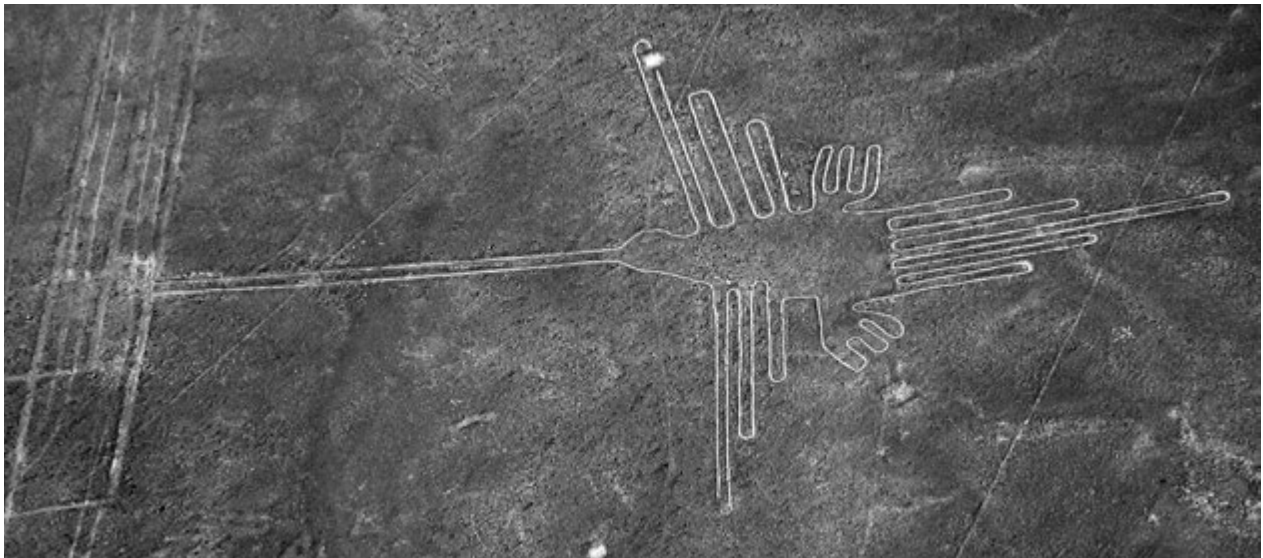


**LILIAN WIESER** was born 1985 in Vienna, Austria. Under her pseudonym “Lilian Boloney” she works as a textile artist in Vienna. She also studies art education at the University of Applied Arts Vienna and the teachers training for mathematics to combine both disciplines. She followed Glaeser with great curiosity on the traces of the Nazcas.

## INTRODUCTION: AN AMAZING WORLD HERITAGE SITE

The pre-Columbian Nazca culture (300 B.C. to 600 A.D.) in the southern Peruvian coastal valleys used to decorate their pottery and embroideries with stylized animal motifs. They are more famous, however, for scraping gargantuan animal depictions into the dry desert ground, often many hundreds of meters across.

These images can only be seen in their entirety from an airplane (Fig.1, Fig.2) and even appear perfectly symmetric from certain angles. This masterful accomplishment has also led to far-out speculations. Some, for instance, have claimed that the Nazca people must have employed hot-air balloons!



*Figure 1: A hundred meter-long hummingbird in the desert.*

Wikimedia. Photo by Bjarte Sorensen

[http://de.wikipedia.org/wiki/Nazca-Linien#mediaviewer/Datei:Nazca\\_colibri.jpg](http://de.wikipedia.org/wiki/Nazca-Linien#mediaviewer/Datei:Nazca_colibri.jpg)



Figure 2: A monkey in the same size. How can such “geoglyphs” be produced? Wikimedia. Photo by Markus Leupold-Löwenthal. <http://de.wikipedia.org/wiki/Nazca-Linien#mediaviewer/Datei:NazcaLinesMonkey.jpg>

The giant animal figures do not appear symmetric from straight above, but rather seem to exhibit perspective distortions as in a skewed photograph. Quite paradoxically, this leads to an appearance of symmetry from certain positions, which some commercial airplanes are fortunate to cross on their flight path.

Without any claim to truth, one may imagine the following story of how these figures came about:

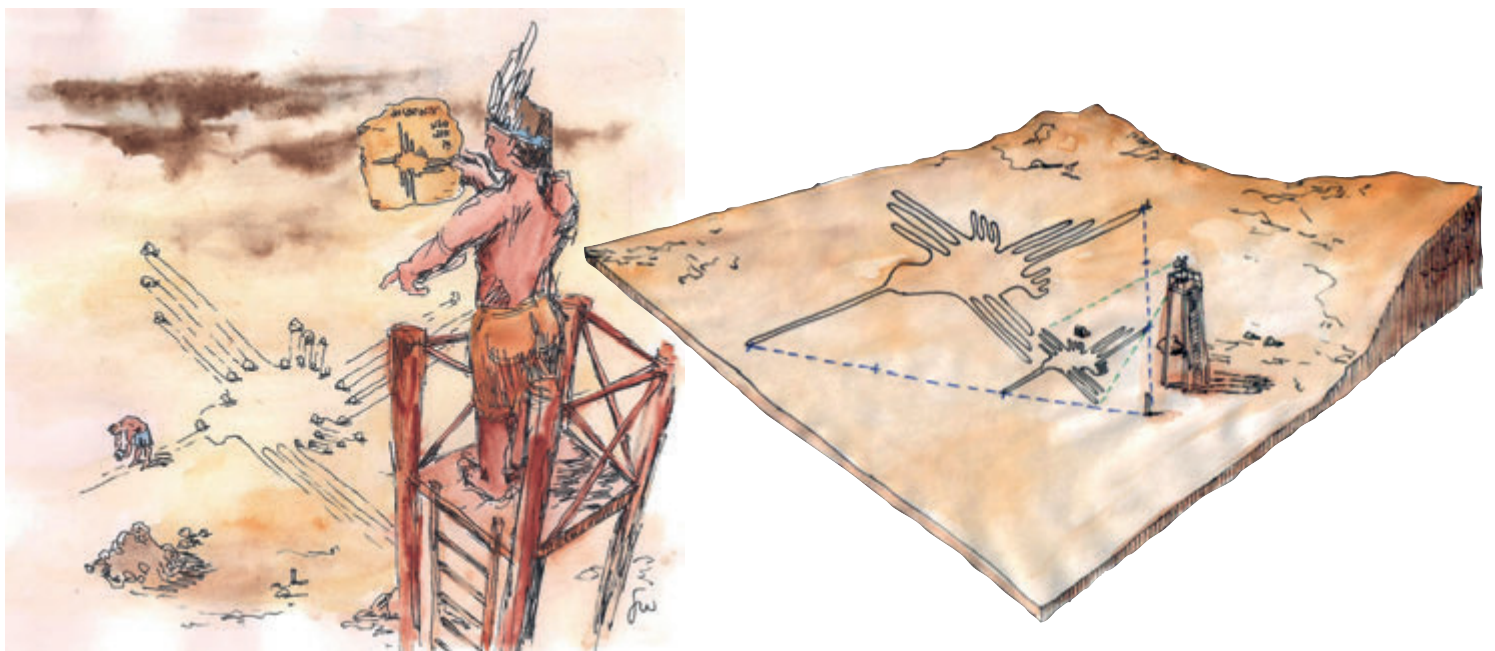


Figure 3: Left: The leader of this huge project tells his assistants where to go to put stones on the ground. Right: The image has to be enlarged afterwards to achieve the giant dimensions we can see today in Nazca. Drawn by Stefan Wirnsperger.

It might have been a high priest standing on a tall and wooden tower (Fig. 3 on the left), holding the symmetric drawing of an animal. He then may have directed his assistants to lay stone blocks to mark a system of lines, that would be consistent with the image from his point of view. His helpers of course would have had to work in quite a distance from the tower, because otherwise, the priest would have needed a “panoramic view”. This would make the animal motif appear symmetric from the low perspective of the tower, while seeming distorted like a skewed perspective view from far above. To build an animal that appears symmetric from a divine point of view – what the Native Americans might have wanted to achieve – the picture has to get enlarged in a certain ratio. Therefore a rope has to be stretched from a fixed center to one of the many contour rocks. Then the length of the rope has to be applied multiple times in the course of the enlargement process (Fig.3 right). This must involve at least two people: one person at the center point of view directing the other at the end of the rope, which ensures the straightness of the ray. At the end of the multiplied distance, the point is marked with another rock. This technique makes it possible



to construct an enormous contour in a relatively short period of time. It may also have allowed the Native Americans to create those gargantuan depictions without having access to today's airplane perspectives.

## FROM THE SQUARE TO THE HUMMING BIRD

Let us make an experiment, and try to act like a high priest (or project leader) with their subjects (co-workers). A school gym is the perfect place to carry out such an experiment (Fig. 4, 5, 6) which we performed as follows:

Parameters and instruments:

- Some rope (about 20 meters)
- a roll of adhesive tape (painters use it), because it is easy to remove from the floor without leaving traces (or chalk, if the experiment is conducted on asphalt)
- a square piece of paper (about 7x7cm)
- at least two persons, although it can also be a group experiment
- a place, where one can reach a higher viewing point and a wide enough space in front of this point (it depends on the dimension of enlargement)

The leader stands at the bottom of a ladder for example and holds the square piece of paper “frontally” in his or her outstretched hands such that the square appears undistorted from one of the “viewing eye” (compare to Fig. 3 on the left, just without the tower, and the co-workers not so far away). Now the co-workers have to react on his or her command. They put some marks at the position on the floor that is “behind” each of the four corners of the square with the adhesive tape. In the next step, the four marked points also need to be connected with the tape (if one is working with chalk, it is possible to use the stretched rope as a ruler, to achieve a straight line). The result on the floor of the gym is an irregular quadrangle or a trapezoid, if two sides of the square were taken horizontal. The leader now replaces the paper square in the hand by a camera which has to be as close to the viewing eye as possible (the back of the camera should be parallel to the former paper position) and takes a picture. Logically, the generated quadrangle will then appear as a square in the picture (but only from the leaders' point of view!).

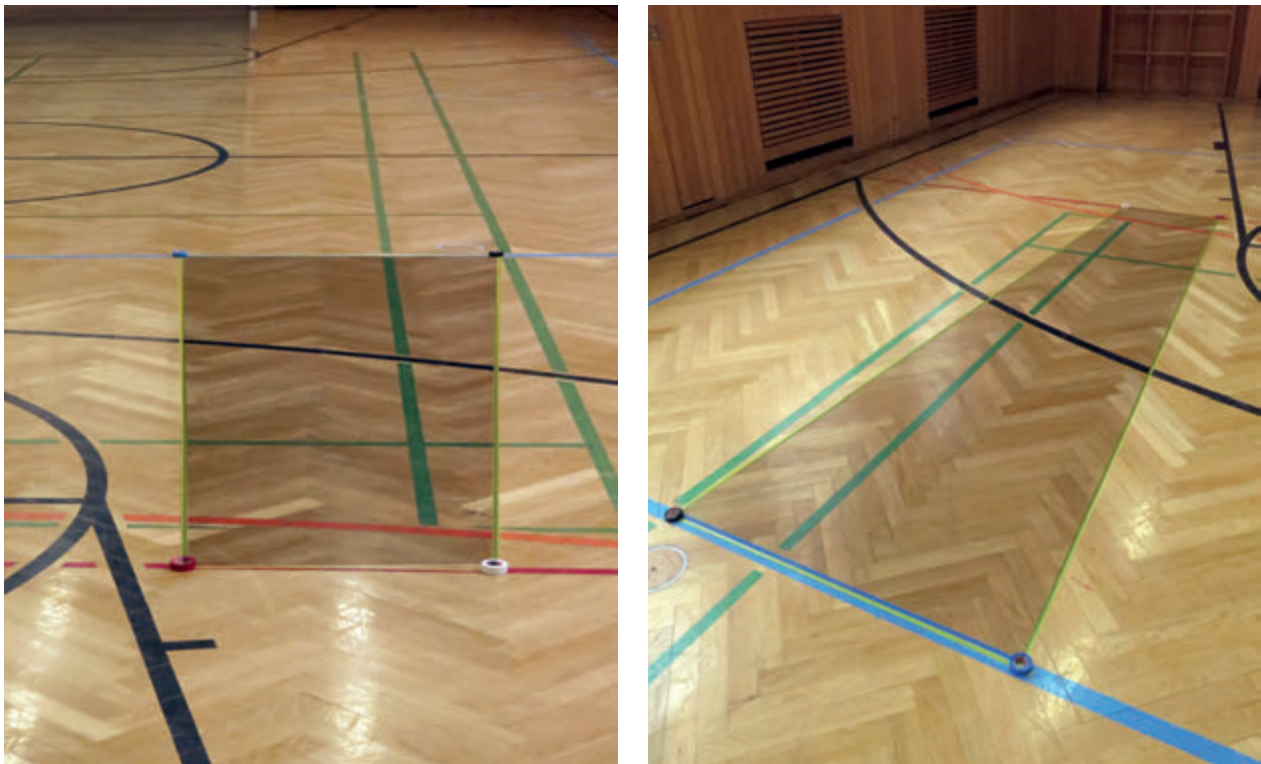


Figure 4: Left: Only from one viewpoint, the quadrangle will appear as a square. Right: If we change the position, the quadrangle has nothing to do with a square.

Photos by Georg Glaeser

## ENLARGE THE QUADRANGLE

In the next step, we want to double (or also triple, or even quadruple if the gym is large enough) the size of the quadrangle. We do this with the “stretch center”  $S$  (Fig. 5), the “original stand point” of the leader. We span a rope from  $S$  to each of the four points  $A, B, C, D$  of the quadrangle, and add this length (or a multiple of it) in the other direction (the enlarged line has to stay straight, of course). Thus we get new points  $A^*, B^*, C^*, D^*$  with twice (triple, etc.) distance from  $S$ , and the new quadrangle is similar, but scaled with a factor of two (or three, etc.).



Figure 5: Scaling of the quadrangle (in this case with factor 2) by adding the rope length from  $S$  to the four vertices  $A, B, C, D$  once more. The eye point  $E$  is also scaled and marked on the ladder. Photo by Georg Glaeser

## CAN THE SCALED QUADRANGLE ALSO APPEAR AS A SQUARE?



Originally, the smaller quadrangle appeared as a square exactly from the leaders' eye point  $E$  (Fig. 5). Because of the scaling, it will now appear in the same matter from the “scaled eye point”  $E^*$ , which is twice as high (or triple, etc.). So, if we now climb up the ladder to the adjacent height and take a picture from there, the enlarged quadrangle will appear as a square. The smaller quadrangle, however, will not be a square anymore (Fig. 6 left).

Figure 6: If we climb up the ladder to the double height, the larger (=upper) quadrangle will now appear as a square. Photo by Georg Glaeser

## WHAT HAS THE QUADRANGLE TO DO WITH THE HUMMING BIRD?

Each Figure can be “polygonized”, like the Native Americans did with the hummingbird, which we can find in the Peruvian desert. So, after the simple case of the quadrangle one has to gain enough practice to conduct a more sophisticated experiment, and create a figure that is more complicated (for example, regular triangles, or regular pentagons etc.). But not only the Native Americans experimented with shapes and perspective. Also the French contemporary artist Georges Rousse applied this phenomenon on local conditions of abandoned houses. In his work it is possible to perceive the principle of a centered perspective in different manners. He might just have created his images to play with mathematical facts, geometrical shapes, color and the local conditions. The Nazcas could have matched each of their images by another special viewing point or viewing direction. Perhaps it was the direction of a divine star. But this is of course pure speculation...

In any case, this is an exercise that teaches us about polygons, standpoints and viewpoints, viewing angles, etc. It does so both theoretically and practically. The experiment is very easy to realize in a school gym or any other place which provides a high vantage point and enough space. The tools necessary for this exercise are regular household items. As a simple experiment, this project allows us to connect to several disciplines, such as mathematics, art, geography, history and archeology.

## REFERENCES

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- Glaeser, Georg. 2012. *Geometry and its Applications in Arts, Nature and Technology*. Vienna/New York
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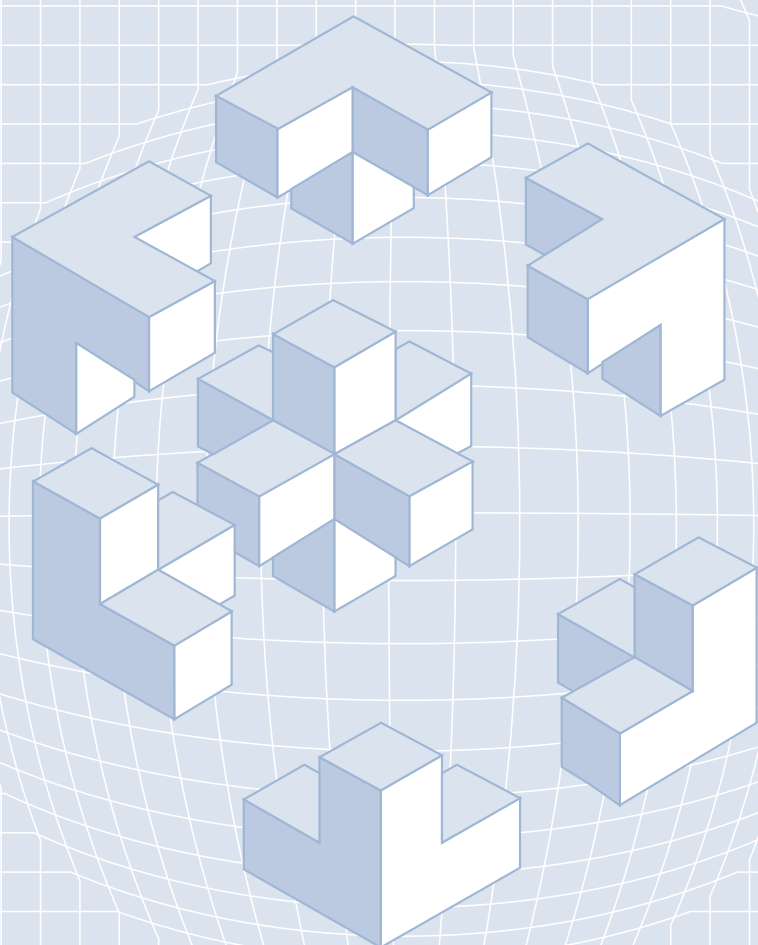


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KRISTÓF FENYVESI - ILDIKÓ SZABÓ

BUILD A GEODESIC DOME AND  
DESIGN ANAMORPHOSIS WITH  
THE EXPERIENCE WORKSHOP -  
INTERNATIONAL MATH-ART  
MOVEMENT



TRICKY STRUCTURES, PLAYFUL  
PERSPECTIVES



**KRISTÓF FENYVESI** is a postgraduate researcher of contemporary culture at University of Jyväskylä's Department of Art and Culture Studies; director of community events at the world largest art and mathematics community, the Bridges Organization ([www.bridgesmathart.org](http://www.bridgesmathart.org)); chief executive officer of International Symmetry Association ([www.symmetry.hu](http://www.symmetry.hu)) and director of Experience Workshop – International Math-Art Movement for Experience-centered Education of Mathematics ([www.experienceworkshop.hu](http://www.experienceworkshop.hu)). His articles have appeared on fora such as *The Notices of the American Mathematical Society*, *MAA Focus – Newsmagazine of the Mathematical Association of America* and the *Bridges Conference Proceedings*. Organizer of conferences, math-popularization events, curator of exhibitions, editor of several textbooks and anthologies. His main areas of research are mathematical connections in arts, culture and education; interdisciplinary aesthetics and philosophy.



**ILDIKÓ SZABÓ** is a mathematics-physics teacher at ANK-School, Hungary and one of the founding members of the Experience Workshop – International Math-Art Movement ([www.experienceworkshop.hu](http://www.experienceworkshop.hu)), taking part in national and international education development projects as the Experience Workshop's pedagogical chief, as well as being the facilitator of creative school days of the Experience Workshop in Hungary and abroad. She has been the proud recipient of a number of professional awards, publishing her results in scientific articles, conferences and popular articles.

## 1 BUILD A GEODESIC DOME FROM NEWSPAPER!

### 1.1 Introduction

The first geodesic dome was a planetarium built according to Walter Bauersfeld's plans for the Zeiss Company in Jena in 1922. Some decades later Richard Buckminster "Bucky" Fuller, an American architect and engineer popularised the special structure of the geodesic dome throughout the world. According to his plans, a geodesic dome was designed to cover the American pavilion for the World Fair in Montreal in 1967. The building can still be seen: its diameter is 80 m and it is 65 m high.

- Compare these measurements with your own height!

Figure 1: The American Pavilion covered by a geodesic dome built at the Montreal World Fair according to Richard Buckminster Fuller's plans.

The name of the geodesic dome comes from the





Greek word Geodos meaning Earth.

- Take a closer look at Figure 1 and Figure 2 and read the captions! Do some Internet research on the content of the pictures. Discuss why Fuller gave the famous pavilion in Montreal and similar buildings the name of geodesic that is, "Earth-like" structure!



Figure 2: A stamp with Fuller's portrait and his creations around him, a photo of Fuller at the World Fair in Montreal in 1967 standing inside the structure designed by him, a photo of the transportation of a geodesic dome by a helicopter, Fuller showing the tensegrity structure and the so-called Dymaxion map.

- More photos and further information on Fuller's career and other work can be found on the following website:  
<http://arttattler.com/architecturebuckminsterfuller.html>



## 1.2 Discover the shape and analyse a digital model!

A geodesic dome is a spherical or partial-spherical shell structure or lattice shell based on a network of great circles (geodesics) on the surface of a sphere. The geodesics intersect to form triangular elements that have local triangular rigidity and also distribute the stress across the structure. (cf. [http://en.wikipedia.org/wiki/Geodesic\\_dome](http://en.wikipedia.org/wiki/Geodesic_dome)).

Our aim is to build the structure of a large dome from pipes made from newspaper. The dome should be large enough for a person to fit under it.

Before starting to build the structure, with the help of the shareware software *Poly*, analyse the digital model of the structure intended to be built!

The *Poly* software is available at: <http://www.peda.com/poly/>

With the *Poly* software several geodesic structures can be studied. First choose the *Geodesic Spheres and Dome* option from the roll-down menu of the software and select some of the *icosahedral geodesic spheres* in the sub-menu. Slowly turning the model, the structures and the edges of the body subdivided by two-dimensional shapes can be closely observed. Turning the model around at speed gives the impression that a sphere and not a polyhedron is rolling in front of you. This is caused by the spherical structure of geodesic domes.

*While experimenting with the software, observe different geodesic structures!*

*In what ways are they similar to one another and how do they differ?*

*What do you think, what the frequency number refers to?*

The construction of Fuller's geodesic dome is based on the spatial geometric shape, called *icosahedron*. If you take a closer look at Figure 3, you can see that each edge of the *icosahedron* is of the same length, triangles being components of the structure are equal in size. The icosahedron is composed of 20 identical equilaterals and a sphere can be circumscribed around the structure.

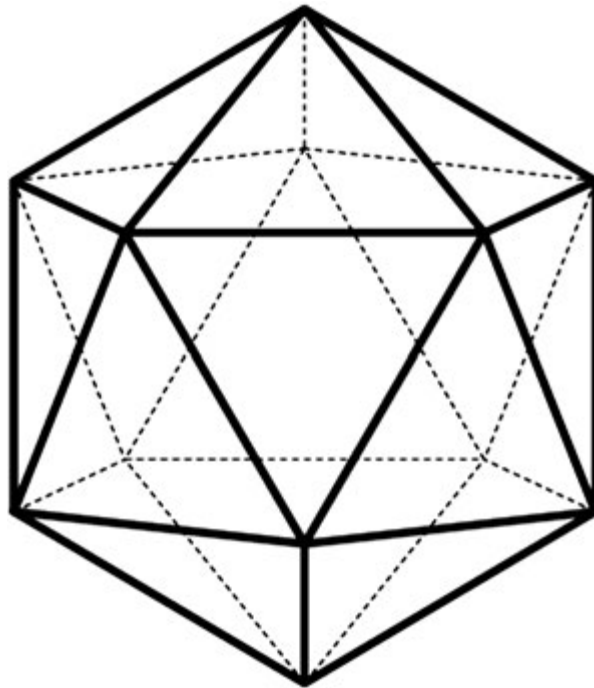


Figure 3: The icosahedron, the origin of geodesic domes.

Features concerning the edges of the geodesic dome are denoted by the frequency number. Because of the equal length of edges the frequency number of a geodesic dome generated from a regular icosahedron is 1.

In order to increase the frequency of a geodesic structure, two actions are required:

Firstly sub-divide the triangular shaped sides of the icosahedron into smaller triangles and then project out the vertices of the triangles on the imaginary sphere circumscribed around the icosahedron as in figure 4:

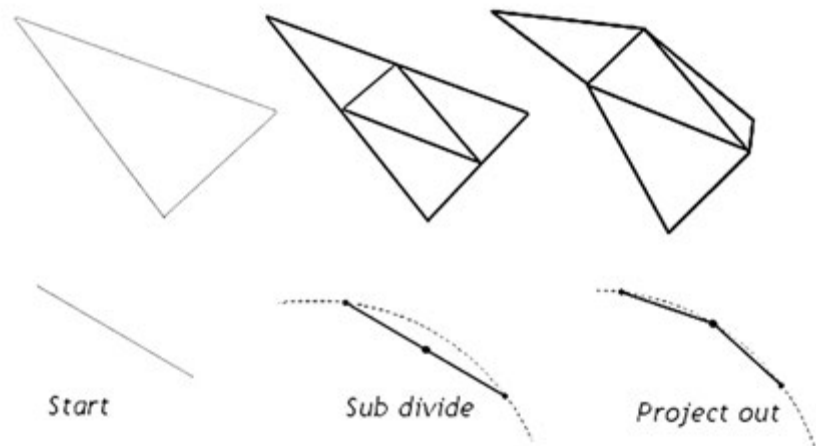


Figure 4: Increasing the frequency number in the geodesic dome based on a regular icosahedron.  
(Source: <http://www.geo-dome.co.uk/article.asp?uname=domefreq>)

A geodesic sphere with higher frequency number can be generated so that we subdivide the face into smaller equilaterals. Figure 5 shows geodesic spheres based on an icosahedron with frequency numbers of 1, 2, 3, 4 and their construction:

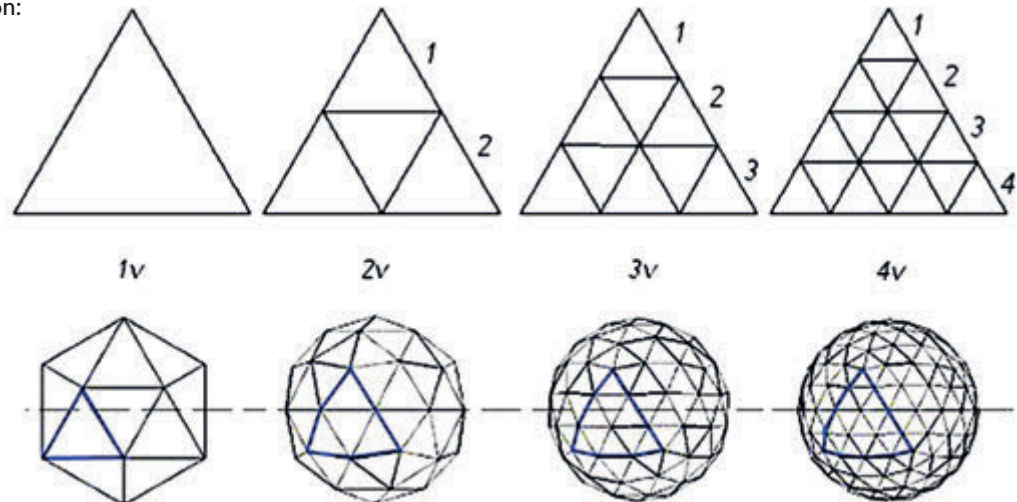


Figure 5: Geodesic spheres based on an icosahedron with frequency numbers 1, 2, 3 and 4 and their construction.  
(Source: <http://www.geo-dome.co.uk/article.asp?uname=domefreq>)

What is the frequency number of the geodesic spherical house in Figure 6? Would you move in? What are the advantages and disadvantages of a geodesic spherical house? Discuss!



Figure 6: Geodesic spherical house (Source: <http://www.geo-dome.co.uk/article.asp?uname=domefreq>)

If you want to determine the frequency number of a geodesic sphere then the best way is to find the centre of an arbitrary pentagon on its surface and count how many struts are there between this centre and that of the adjacent pentagon (Figure 7).

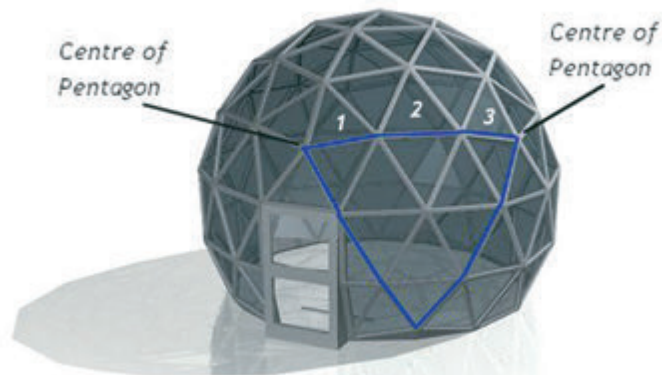
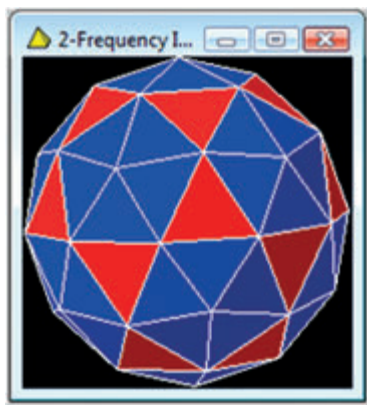


Figure 7: Determining the frequency number of the geodesic sphere. (Source: <http://www.geo-dome.co.uk/article.asp?uname=domefreq>)



With the help of the software *Poly*, geodesic spheres with different frequency numbers can be studied (Figure 8).

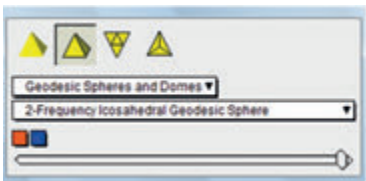
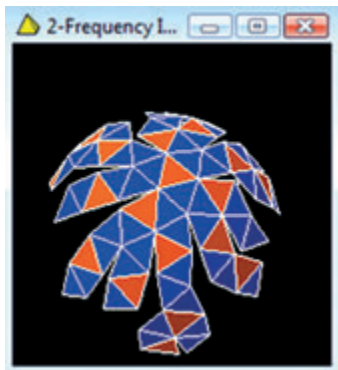


Figure 8: Visualising a geodesic sphere with a frequency number 2 in the software *Poly*.



Moving the cursor you can even unfold solids. Gradually you leave space and enter a plane (Figure 9).

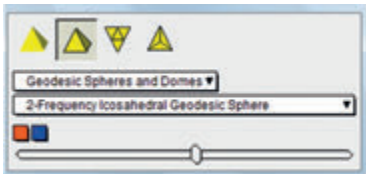
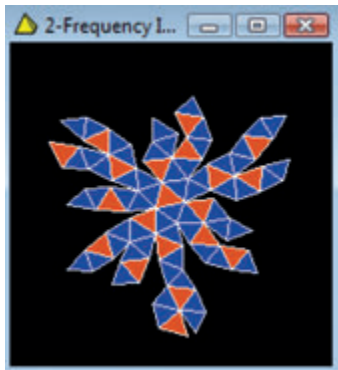


Figure 9: Unfolding the geodesic sphere by moving the cursor. We are gradually leaving space and entering into a plane



The cursor can be moved so that the flattened hollow solid, the net, can be visualised (Figure 10). Observe that the smallest two-dimensional figure unit into which the surface is subdivided consists of all regular and isosceles triangles. You can count how many regular and isosceles triangles border different geodesic structures.

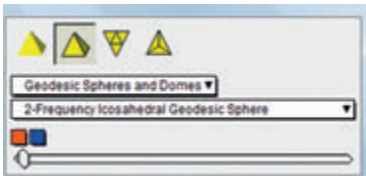


Figure 10: A geodesic sphere unfolded into a plane having a frequency number 2.



Rotate the structure in each case then visualise the net and observe different solids.

Finally find the model called *2-Frequency Icosahedral Geodesic Sphere*. For the sake of simplicity we are going to build this now. Later you can try to build more complex structures!

**1.3 Build a geodesic dome from newspaper!**

As you can observe and count on our digital model generated in the software Poly you will need 35 pipes 65 cm long (blue in the pictures) and 30 pipes 58 cm long (red in the pictures) in order to build the geodesic dome shown in the figure below (Figure 11).

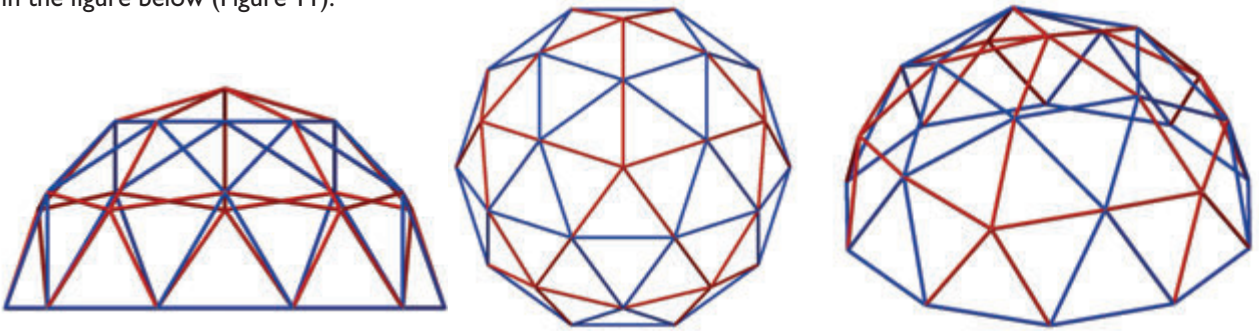


Figure 11: Geodesic sphere to be constructed based on an icosahedron with a frequency number 2 from side-view (to the left), viewed from above (in the middle), in the space (to the right). (Computer graphics are made by László Vörös)

Using wooden skewers and newspaper, make paper pipes for the frame structure of the dome according to the sequence of Figure 12. As Figure 12 shows, place the newspaper on the table. Take any skewer, place it perpendicular to the diagonal of a turned rectangular sheet of paper in the corner of the newspaper. Roll the turned newspaper tightly around the skewer paying attention that the skewer remains perpendicular to the diameter. When the newspaper pipe is ready then glue the corner of the newspaper at the screwed end with some transparent glue so that the paper will not unroll.



Figure 12: Making paper pipes for the frame structure of the dome using wooden skewers and newspaper.

Measure off and mark the required length on the completed newspaper pipes so that the required length remains 1–1 cm longer and cut off the unwanted bit at the end of the paper pipes (Figure 13). This extra length is required as the pipes will be joined by a stapler at the junctions during the construction where rods constituting the structure of the frame join together, and by leaving excess it is easier to join the pipes with a stapler during the construction.

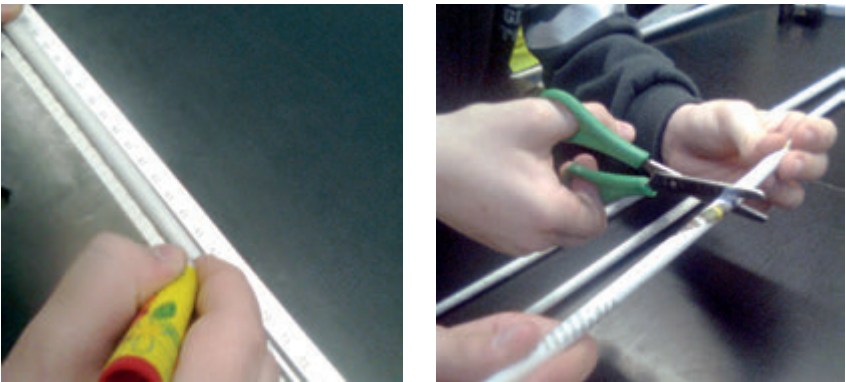


Figure 13: Marking and cutting off the pipes made of newspaper.

*Study the base of the dome!* You can see that the base of the dome is an area enclosed by a regular decagon. Place 10 blue (long) pipes on the ground as in Figure 14.

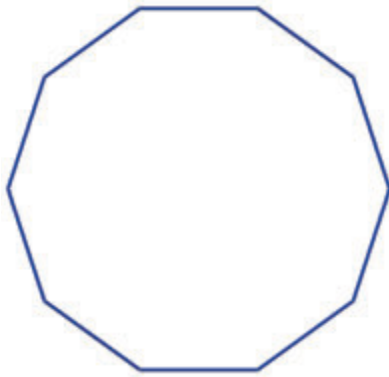


Figure 14: Base of a geodesic dome

Calculate the size of the angles of the regular decagon!

It is worth making 10 templates of the angle of 144 degrees and with their help join the pipes together (forming the required angles) in order to succeed in joining precisely.

After bordering the area around with pipes staple them together with a stapler and in this way the decagon constituting the base is ready (Figure 14).

In the next steps follow the pictures below. Join pipes with a stapler, or if possible, you can use glue to seal the joining points. The construction phases (shown in top view) are as follows (figures 15-21):

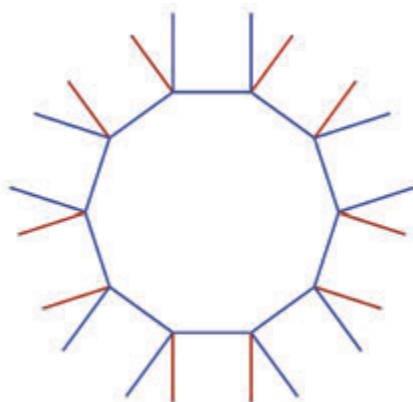


Figure 15

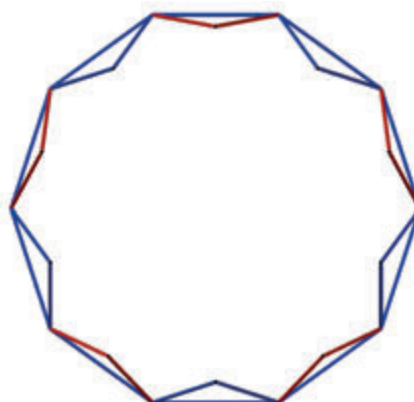


Figure 16

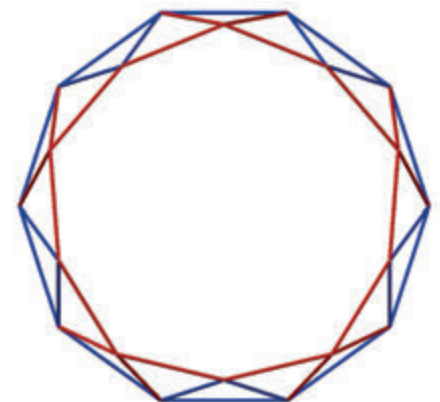


Figure 17

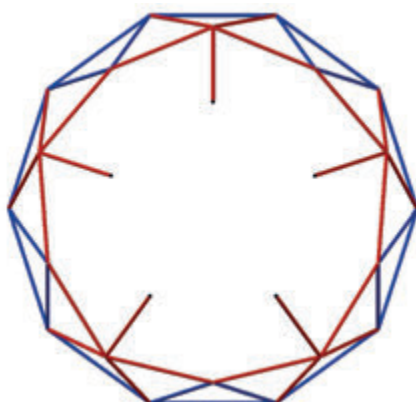


Figure 18

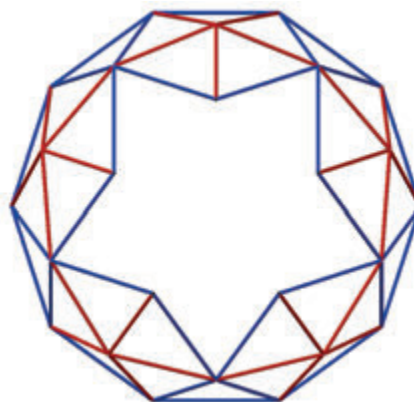


Figure 19

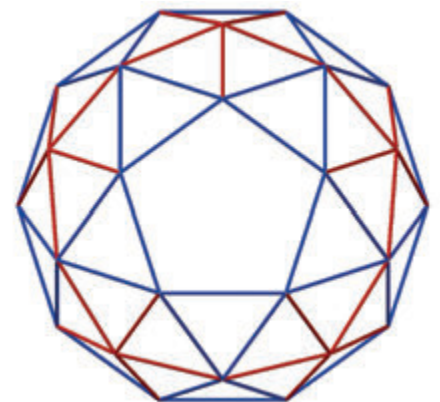


Figure 20

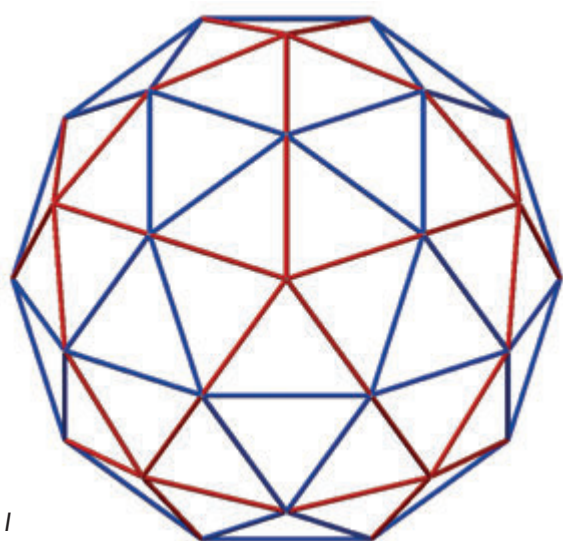


Figure 21

Figure 15-21: Construction phases of the dome shown from top view (computer graphics are made by László Vörös, Ph.D.)

Three-dimensional figures of phases of the construction of the dome are as follows (Figures 22-28):

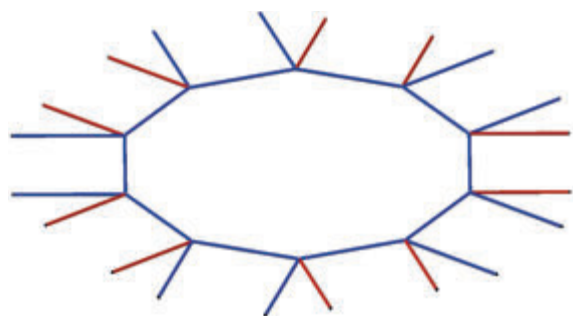


Figure 22

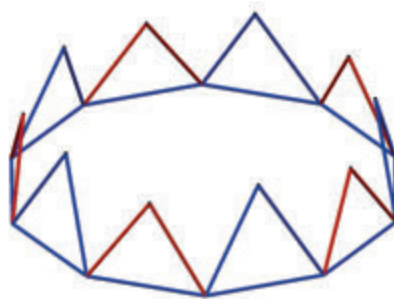


Figure 23

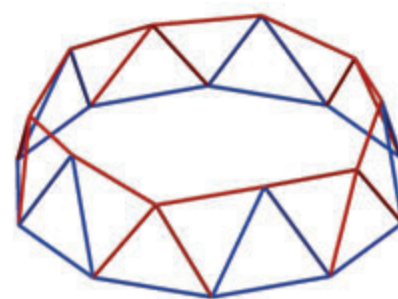


Figure 24

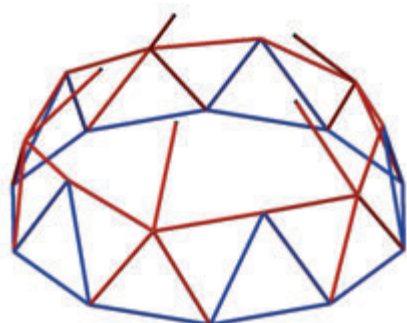


Figure 25

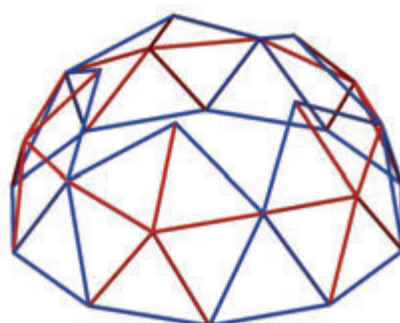


Figure 26

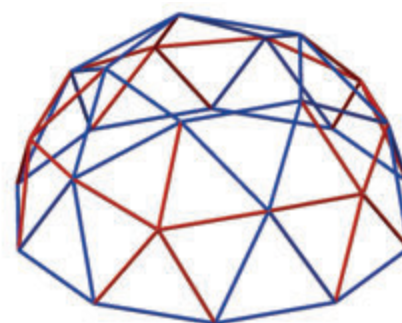


Figure 27

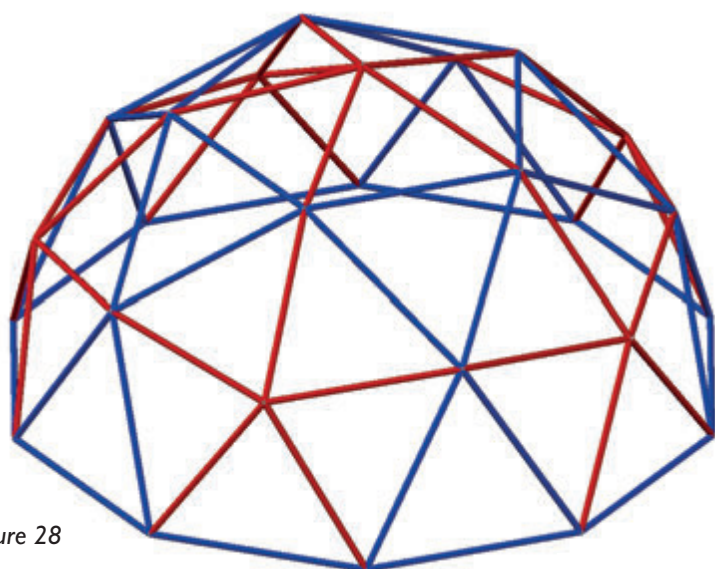


Figure 28

Figure 22-28: Three-dimensional figures of phases of the construction of the dome (computer graphics are made by László Vörös)





Figures 29-33: Photos of the construction process in a classroom

## 2 DESIGN ANAMORPHOSIS IN THE CLASSROOM!

Generating optical illusions and other visual effects through playful creative activities offers considerable potential to arouse and increase interest in mathematics and other sciences, such as physics. Several visual effects are based on phenomena describable from a mathematical point of view, generated by algorithmic operations that can be formalized. Several artworks generated with mathematical algorithms are in the repertoire of the travelling exhibit of the Experience Workshop Math-Art Movement. For instance, István Orosz's or Jan W. Marcus' anamorphic artworks can bring the pedagogical value of anamorphoses, this specific visual effect to teachers' attention.

Anamorphosis means a distorted projection or perspective requiring the viewer to place a mirror on the drawing or painting, or occupy a specific vantage point to reconstitute the image.

Anamorphosis opens up the possibility of discovery in many areas. Hereby, we list only some of them hoping that they will peak your interest and encourage you to explore further. Some areas in which anamorphosis can be implemented in mathematics: studying the location in the Cartesian coordinate system and with polar coordinates; one-to-one correspondence (correspondence of cells) of different reference systems; measuring angles; representation and division of concentric circles into equal sectors, etc; in physics: concepts belonging to geometric optics, such as light sources, refraction, reflection from different surfaces, etc; in biology: biology of vision and biology of the eye, etc.

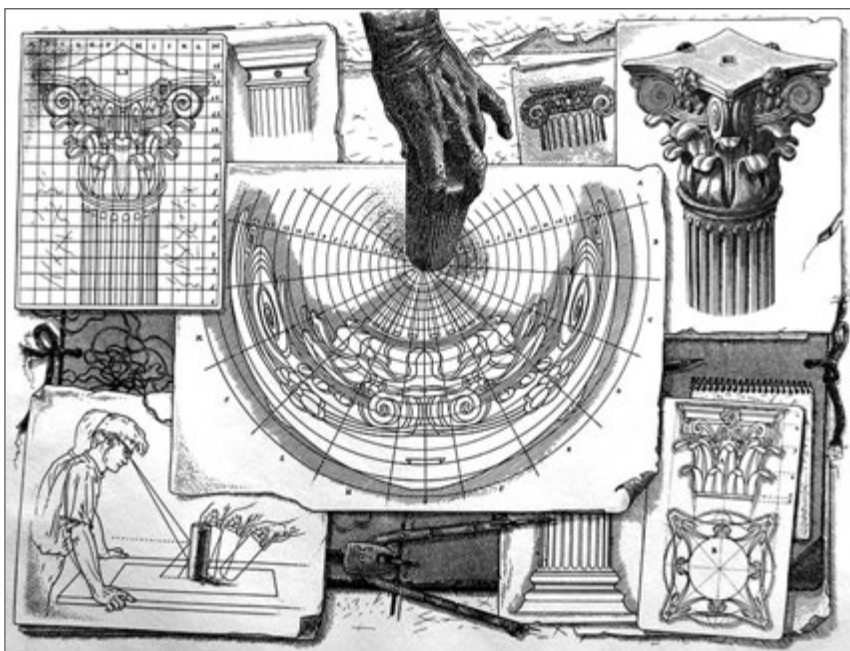


Figure 34: Artworks by István Orosz. Left to right: Anamorphic eye by István Orosz (to the left), and Construction of the anamorphosis by István Orosz (to the right).

- First activity: read the distorted text on the logo



Figure 35: Anamorphic logo of Experience Workshop

#### Observations

- Reading: read from left to right and you are required to imagine the mirrored image of the letters in order to be able to read the text
- The distorted projection is situated in the shape of an almost regular half circle on a paper, etc.

- Second activity: play with mirrors

Try to read the text on the logo with the help of different shaped mirrors.

#### Observations

- By using a mirror it is easier to read the text
- The shape of the mirror you use does matter!
- With a mirror of cylindrical shape placed at the right place (the center), the picture leads to a more precise reading of the text
- Formulating the hypothesis: generating the image with a cylindrical mirror, the hidden text can be made instantly legible.



- *Third activity: make a cylindrical mirror*

Making a cylindrical mirror is very simple: stick mirror film sold in stationary shops to a paper cylinder of the adequate radius (e.g. use a paper cylinder from a kitchen towel roll). Placing the cylindrical mirror on the half circle part of the smooth paper sheet, the picture shown in Figure 35 can be seen with legible text.

In order to create the picture used in the list of activities mentioned above the software which generates the anamorphosis is free to download from <http://www.anamorphosis.com/software.html>.

In this way illustrations used during activities can be generated by using any picture in advance.

- *Fourth activity: observe the technology of making anamorphosis! Find correspondences between networks while generating anamorphosis!*

Everybody gets a sheet of paper on which networks are previously drawn for designing anamorphosis according to the figure and description below. Added lines drawn in red and blue are not in the figures so that students can discover the relationships themselves.

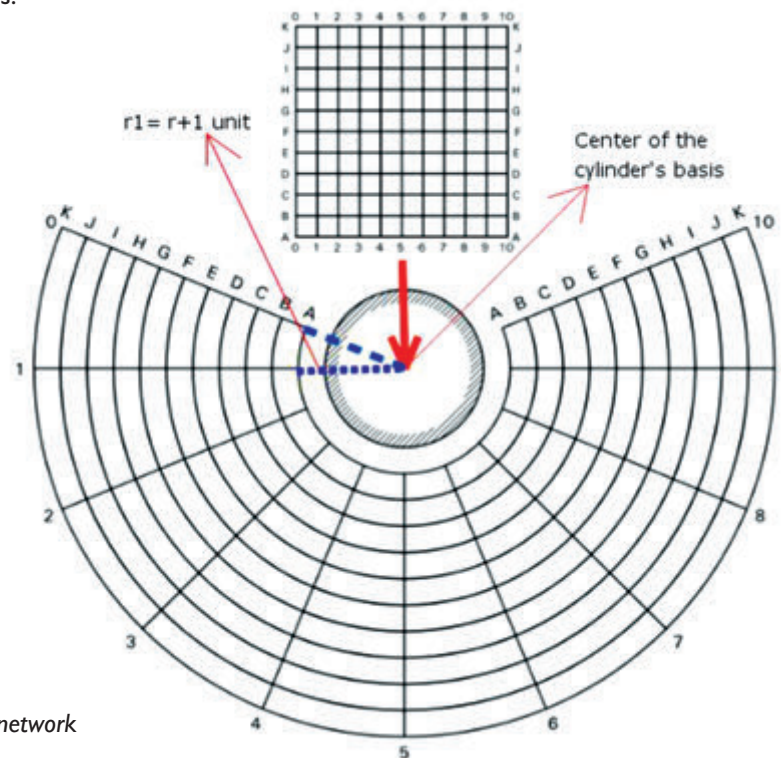


Figure 36: A structure of the network for designing anamorphosis

By drawing the correct lines students can easily design a detorted network for the original network: by drawing circles concentric to the circle constituting the base of the cylindrical mirror and then dividing them into angles delimited by sides measured 22.5 degrees with a vertex being in the centre, the already detorted cells are formed. The red arrow points to the centre of the base of the cylinder, the blue broken lines are the radii of the first circle drawn around the cylinder concentric to the circle of its base. The letter 'r' denotes the radius of the circle of the base of the cylinder (see Figure 36).

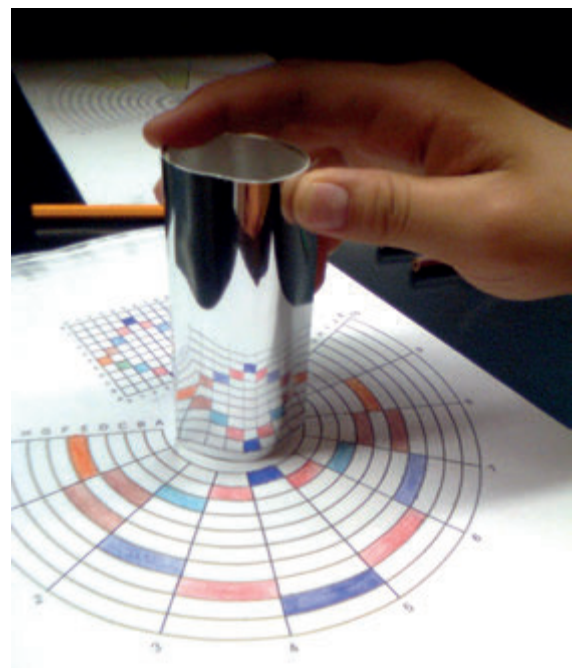


Figure 37: Placing the cylinder mirror



- *Fifth activity: use laser*



Fix a laser light source on a retort stand. Be careful when using the laser not to shine it directly into your or anybody else's eye. You should set up the laser so that it equates with your sight of view as if you were looking directly at the paper from above. Students can observe where points corresponding to each other get to during the reflection of light: the point of light on the cylinder, and its image on the net (correspondence of cells). The same – keeping safety at the forefront of your mind and with a teacher's constant presence – should be tried also from the children's perspective.

Figure 38: Use laser!

- *Sixth activity: design anamorphosis*

On a regular square network, students can make their own creations and by copying the corresponding cells on the detorted structure of the network the detorted picture can be generated. Cells determined by letters and numbers help again in mapping. Then placing the cylindrical mirror on the marked circle the designed creation can be viewed immediately.

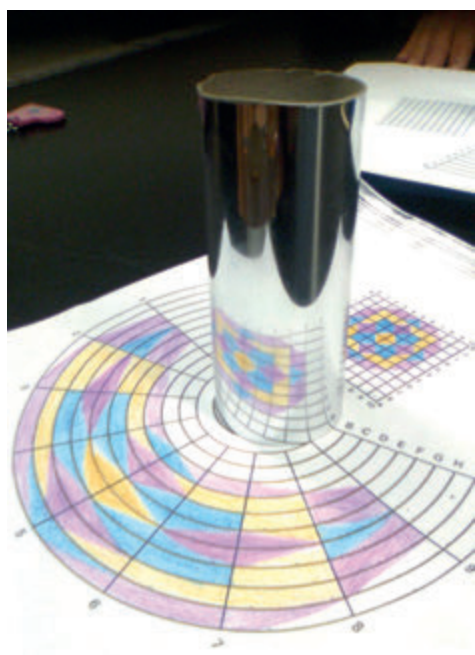
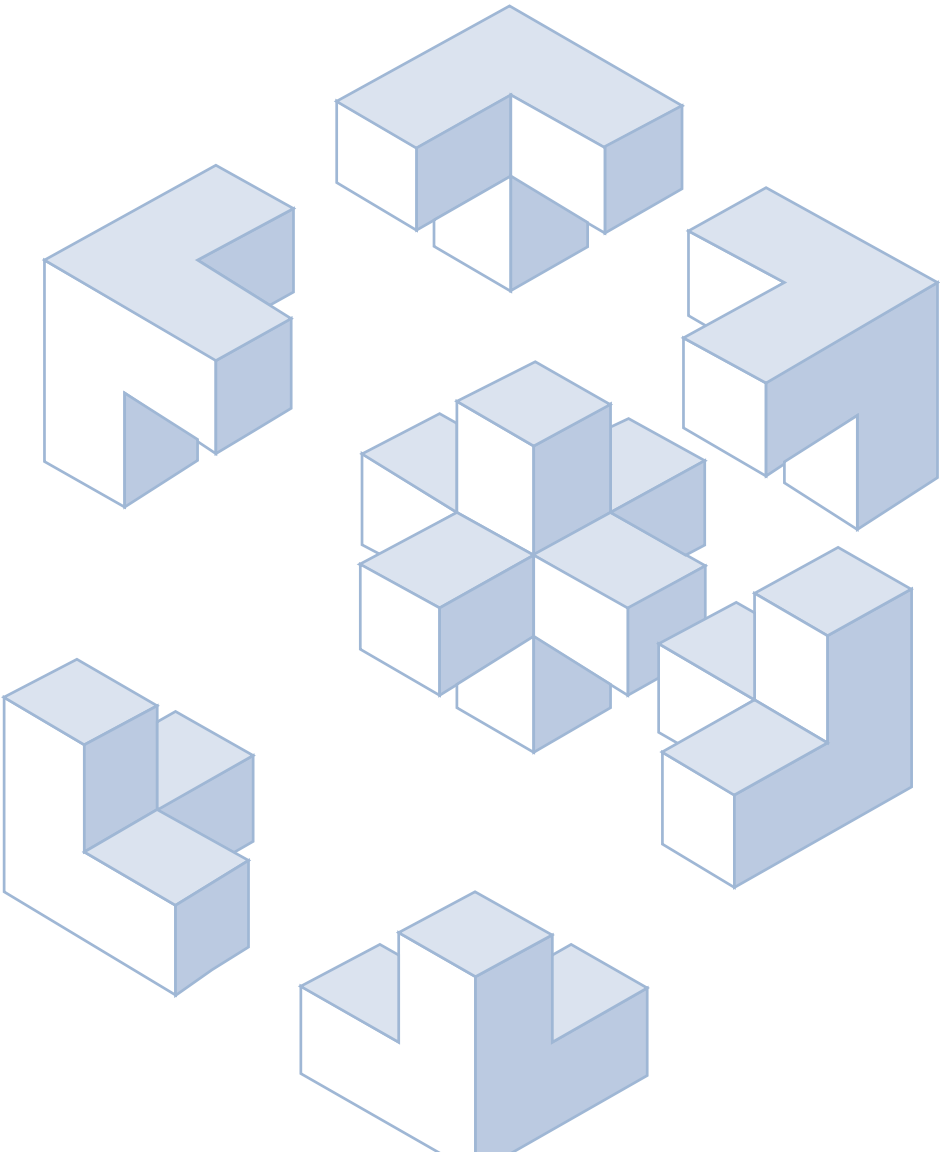


Figure 39: Design anamorphosis!

- *Seventh activity: design your own network*

Choose a paper roll and stick a mirror film on its surface. Draw the structure of a network including squares and denote cells by numbers and letters according to the description in the fourth game. Place the cylindrical mirror on a sheet of paper and draw around its base. Then, according to the description outlined above, draw the other network structure in the shape of a circle. Being placed into it, the detorted picture will be seen as an image in the cylindrical mirror in a predetermined form.



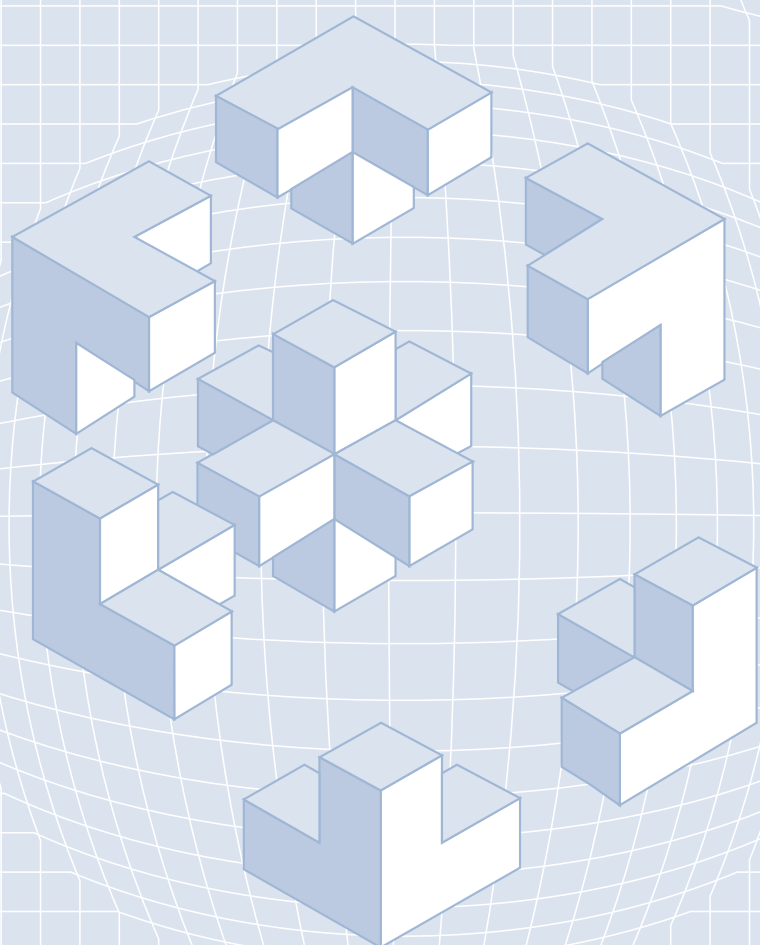


VISUALITY &  
MATHEMATICS

EXPERIENTIAL EDUCATION  
OF MATHEMATICS THROUGH  
VISUAL ARTS, SCIENCES  
AND PLAYFUL ACTIVITIES

DIRK HUYLEBROUCK

SPATIAL  
FRACTAL WORKSHOPS



TRICKY STRUCTURES, PLAYFUL  
PERSPECTIVES



**DIRK HUYLEBROUCK** spent eight years at universities in the Congo until a diplomatic incident between Belgium and President Mobutu of Congo interrupted his stay. He went to the University of Aveiro in Portugal and the European Division of Maryland University, until the majority of his American (military) students went to Iraq. He returned to Africa, to Burundi, but only for three years, because of the genocide in neighboring Rwanda. In 1996, he finally consented to teach at the Faculty of Architecture of the KU Leuven (Belgium). Fortunately, he can still escape abroad, as he has edited the column “The Mathematical Tourist” in “The Mathematical Intelligencer”, since 1997. However, he may soon have to flee abroad again, as he has become increasingly (in)famous due to his work in finding errors in well-known artwork such as in the Belgian Atomium landmark, in the work of Leonardo da Vinci, in Michelangelo’s The Forbidden Fruit, in Van Eyck’s Mystic Lamb, and, most recently, in Norbert Francis Attard’s Fibonacci sculpture.

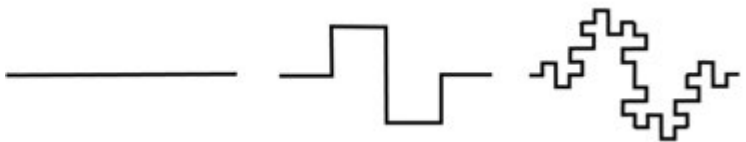


# FRACTALS

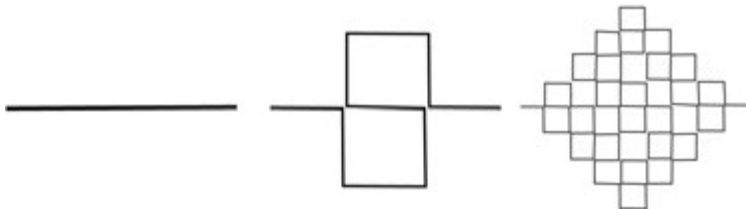
Fractals today are gathering somewhat of a following outside of the mathematical world, this is not really surprising as they are intriguing structures repeating the same ‘base’ form, sometimes even to infinity. Thus fractals seem to have many members just as in nature a cloud, a fern leaf or a cauliflower has smaller and smaller parts similar to the original. In popular mathematics, fractals are much admired, probably because they make graphics pleasing to the eye as evidenced by their use in psychedelic MTV-videos. The notion, however, is over a hundred years old, and it is thanks to the Frenchman Benoit Mandelbrot (who emigrated to the US) that this mathematical subject escaped the academic ivory tower and received the attention it deserved.

The ‘capriciousness degree’ of a curve, or of a surface or of a spatial volume, is called its ‘fractal dimension’. If it is known how many new elements are used repeatedly to form a new figure, and if the scale factor indicates the number of these new parts needed to divide one unit length, then the fractal dimension given the number of new elements = (scale factor)<sup>dimension</sup>.

For example, a line has dimension 1 since  $3 = 3^1$ : if the unit length is divided into 3, then 3 parts fill one unit length. A flat surface has dimension 2,  $9 = 3^2$ , because if the unit length is divided in 3, then 9 new squares fill up a square whose side is the unit length. Finally, a spatial volume has dimension 3, because  $27 = 3^3$ , as 27 cubes fill a cube whose side were divided into three. For the fractal curves shown in Figure 1, the dimension numbers lie between 1 and 2.



A fractal curve with dimension 1.5 because the scale is 4 and 8 times as many parts are used in each step:  $8 = 4^{1.5}$ , because  $8 = (\sqrt[4]{4})^3 = 4^{3/2}$ .



A fractal curve with dimension 2, because the scale is 3, and 9 pieces are used:  $9 = 3^2$ . Knitting a jersey this way yields perfectly insulated clothing.



The dimension of the so-called Koch fractal is approximately 1.26 for the scale is 3 and there are four times as many parts used. A calculator helps to confirm this:  $4 \approx 3^{1.26}$ .

Figure 1: Some fractal drawings and their dimensions.

The mathematical theory of fractals uses complex numbers and more involved set-ups. An application of the more involved mathematical theories can be found in the study of the biological structure of the human lung. In form, a lung can indeed be considered as fractal. In each alveoli air molecules collide against the inner surface of the lungs: Brownian motion (a principle first formalized by Albert Einstein). This means that they collide in a completely random way. The larger the inner lung surface, the greater the opportunity that the air molecules can enter the blood through the surface.

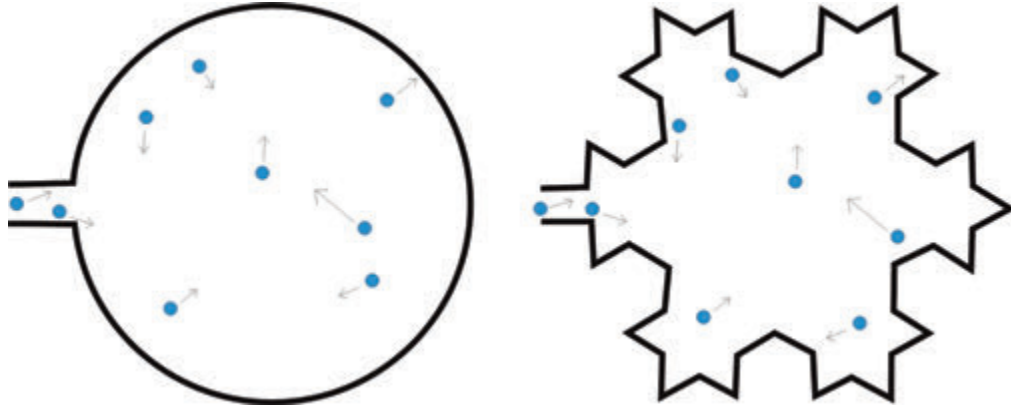


Figure 2: Abstract alveoli, where air molecules in a Brownian motion collide. One has the shape of a circle (that is, of a sphere, in three dimensions) and the other of a Koch fractal, having a larger internal colliding surface.

If the inner surface is too irregular, it could happen that some parts of the area are rarely touched by the molecules, because they lie “around the corner”. Mathematician Tom Wolf demonstrated that surfaces exist with a fractal dimension greater than 2 that are accessible to Brownian motion (fortunately, because this is why we can breathe). Princeton’s IAS Jean Bourgain established a formula proving that the dimension of such an accessible area should be less than 3.

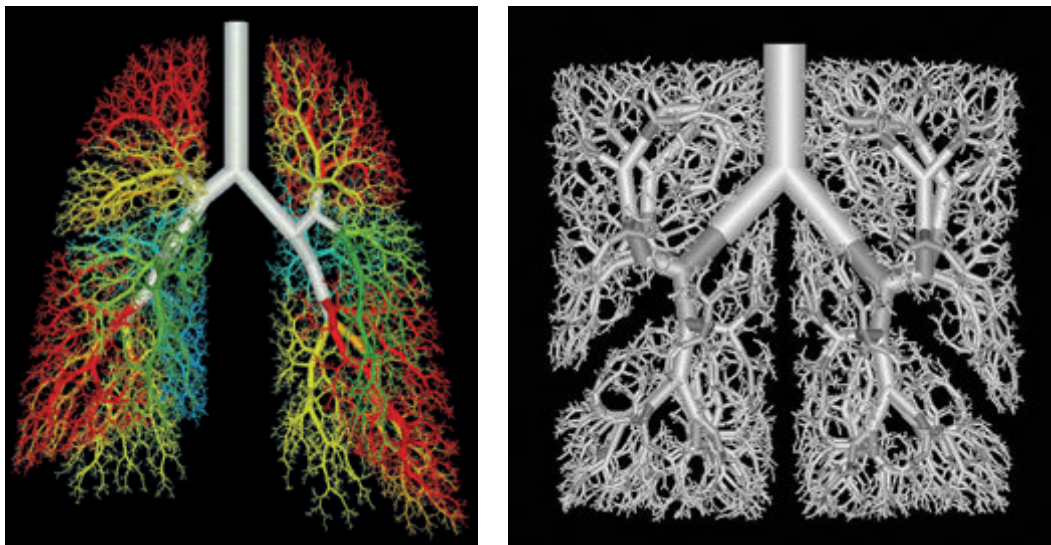


Figure 3: Models of lungs based on fractals; a realistic picture (left) and an imaginary one (right), called ‘squaring the lung’ (Ryuji Takaki, Tokyo, Japan).

Another popular series of applications can be found in ethno-mathematics. Ron Eglash wrote a complete book on “African Fractals”, about, for instance, prominent African-American scientist Benjamin Banneker’s so-called ‘quincunx fractal’ (see [4]). He noted that it is not unusual to encounter this fractal pattern in Senegal as a decoration on small leather bags worn around the neck. Ethiopian crosses would be other examples where the mathematical fractal structure is re-discovered. Surely it is correct to state that many African shapes were inspired by nature and thus incorporated its fractal structure.

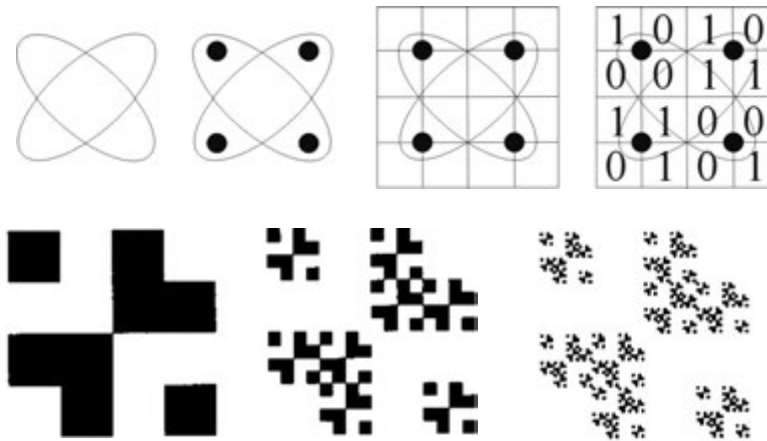


Figure 4: An imaginary example by Paulus Gerdes showing how African sand curves can lead to fractals.



However, though fractals do indeed turn out to be useful in describing natural phenomenon in a very simple mathematical formulation, scientists like David Avnir (The Hebrew University of Jerusalem, Israel) have shown that the fractal property of nature is debatable: very often there are only two or three scale levels, though the possibility of an infinite continuation is an important part of the definition of a fractal (see [1]). In Eglash's 'African fractals', there are often only two or at most three levels. Avnir went so far as to write to Mandelbrot in person objecting to his theories about the fractal interpretation of nature. Their discussion did not reach a definite conclusion though, and so let's finish with a witty remark raised by Patrick Fowler (University of Sheffield, United Kingdom) in private correspondence: "Why isn't there a book about, for instance, 'Catholic fractals', since in some religious art, a saint holds a statue of Mary, holding the child Jesus (see Figure 5)? Is that fractal art too?"

Figure 5: Is this statue a proof catholic artists knew fractals 600 years ago? A saint holds a statue of Mary, who holds a statue of Jesus: is that a 'Catholic ethno-mathematical fractal'?

## SPATIAL FRACTALS

Though fractals have enjoyed great attention in popular science, most examples show planar images or are at best spatial extrusions of planar examples. The image of a so-called fractal tree is an icon in fractal imagination, but in most cases it is a 'flat tree', in two dimensions (see Figure 6). One of the most quoted and also earliest examples of a true spatial fractal is the Sierpinski gasket, named after the Polish mathematician, Wacław Sierpiński. Its planar original is the Sierpinski triangle, having the overall shape of an equilateral triangle, subdivided recursively into smaller equilateral triangles. In space, it corresponds to a tetrahedron, subdivided recursively into 4 smaller tetrahedrons. Variations are possible, of course, such as using a pyramid with a square base and subdividing it recursively into 5 smaller pyramids.

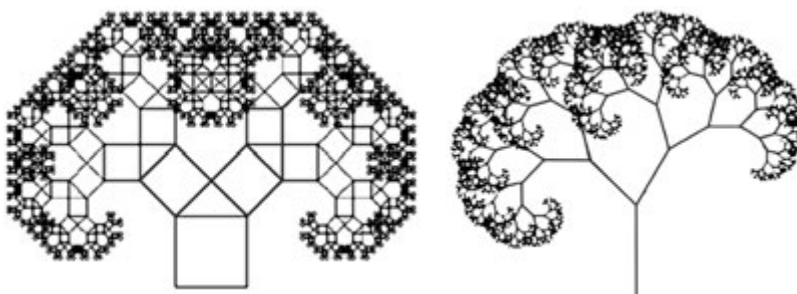


Figure 6: An iconic image: the planar fractal tree (right) based on applying the Pythagorean Theorem (Wikimedia Commons, drawing by 'Fractal World').



The three-dimensional analogue of the Sierpinski triangle already illustrated the kind of problems that can arise with spatial fractals. Indeed, if an initial tetrahedron of side-length  $L$  and surface  $L^2\sqrt{3}$  is subdivided into 4 tetrahedrons of side-length  $L/2$ , the total surface (including the inside areas) will be  $4((L/2)^2\sqrt{3}) = L^2\sqrt{3}$ . Thus, the initial surface remains constant after each iteration. The volume on the other hand decreases by one half with each iteration and thus approaches 0 as the number of iterations increases. Its dimension, using the above formula, is 2, as  $4 = 2^2$  and this corresponds to the dimension of a flat drawing (see Figure 7)

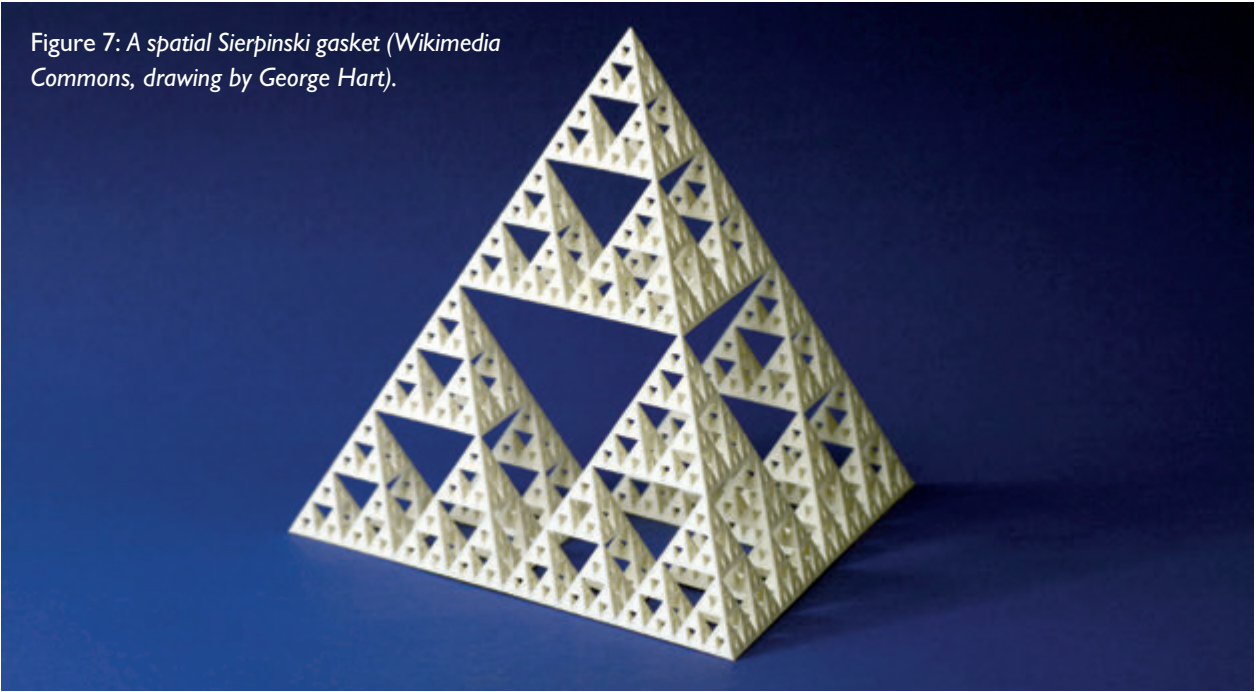


Figure 7: A spatial Sierpinski gasket (Wikimedia Commons, drawing by George Hart).

The fractal tree provides spatial challenges too: the icon drawing is flat, and when making spatial versions, the branches quickly turn into each other, that is, intersect the neighboring branches. This happened to Michiel Duinslaeger, a former architecture student at the Faculty of Architecture of the KU Leuven, Belgium (he graduated in July 2014). He followed an optional math art course and wanted to construct a true spatial fractal tree. When testing some arbitrary lengths and angles, his results were unsatisfactory (see Figure 8).



Figure 8: A spatial fractal tree does exist, as in nature (photo above), but when reproducing it, the 3D drawing quickly becomes overcrowded with intersecting leaves (drawing by Michiel Duinslaeger).

Fortunately, Michiel Duinslaeger shared his math/art course with a group of Serbian students on an EU Tempus project. One of these visiting students, Danka Lučić, of the University of Novi Sad, Serbia, advised Duinslaeger to use a length reduction of  $\frac{1}{2}$ , and branch angles of  $45^\circ$ . She based her suggestion on existing literature (see [2], [5], [6] and [7]). Duinslaeger's design of a 3D fractal tree with 682 pieces and of relatively simple construction will form the basis of the first 3D fractal activity. The second activity builds on the idea of making polyhedra with balloons; this originated in chemistry, but has also become popular in mathematics (see [3]).

## ACTIVITY 1: 3D FRACTAL TREE

### Teacher's guide

**Materials:** Light wood (triplex, or balsa), card board, strong paper.  
Paint.

**Objective:** Students will become familiar with 3D fractals properties, and experience how slowly or quickly each generation grows.

**Vocabulary:** Fractals, 3D space.

**Activity Sequence:**

1. Copy figure 9 in wood (or in cardboard) and paper, or else. What is even easier, use the pieces to be cut out given separately on several pages available for download at this web-address: [vismath.ektf.hu/Dirk\\_Huylebrouck\\_Templates](http://vismath.ektf.hu/Dirk_Huylebrouck_Templates). For instance, use 9mm thick wood for the first generation, 6 mm thick for the second, 3mm for the third, 2mm the fourth, and finally 1mm thin paper.

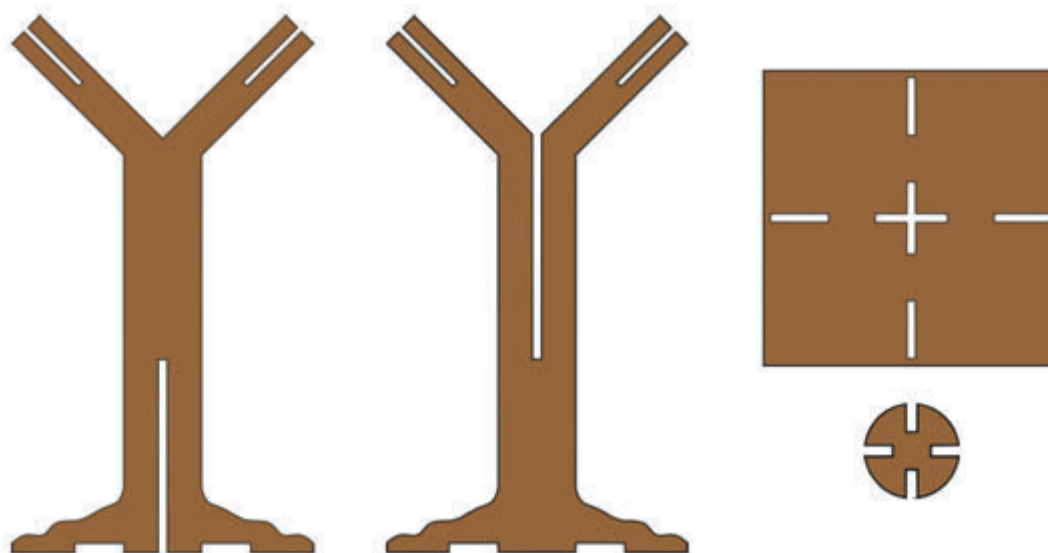


Figure 9: These pieces make the trunk of the tree and the support. They can, for instance, be cut from 9mm thick wood.

Note the width of the vertical opening of the longer piece equals the thickness of the used material for the *previous* fractal generation, while the width of the vertical opening of the shorter piece equals the thickness of the material for its *actual* fractal generation. The other width of the skew openings equals the thickness of the material for the *next* fractal generation.

In the given example 2 wooden pieces make the trunk of the tree, 8 the branches, and finally there are 32 twigs; 128 green leaves and 512 white blossoms complete the tree. This gives a total of 682 pieces but 2 pieces are added for additional support: a large square and a circular shape to keep the trunk firmly together. Here, the lengths of the Y-shaped branches were 18.5 cm, next 9.25 cm, and 4.625 cm and so on.

2. Paint each fractal generation as desired (here we use none, green and white).
3. Have the students put the tree together. And most of all, be patient.
4. What is the fractal dimension of the tree?

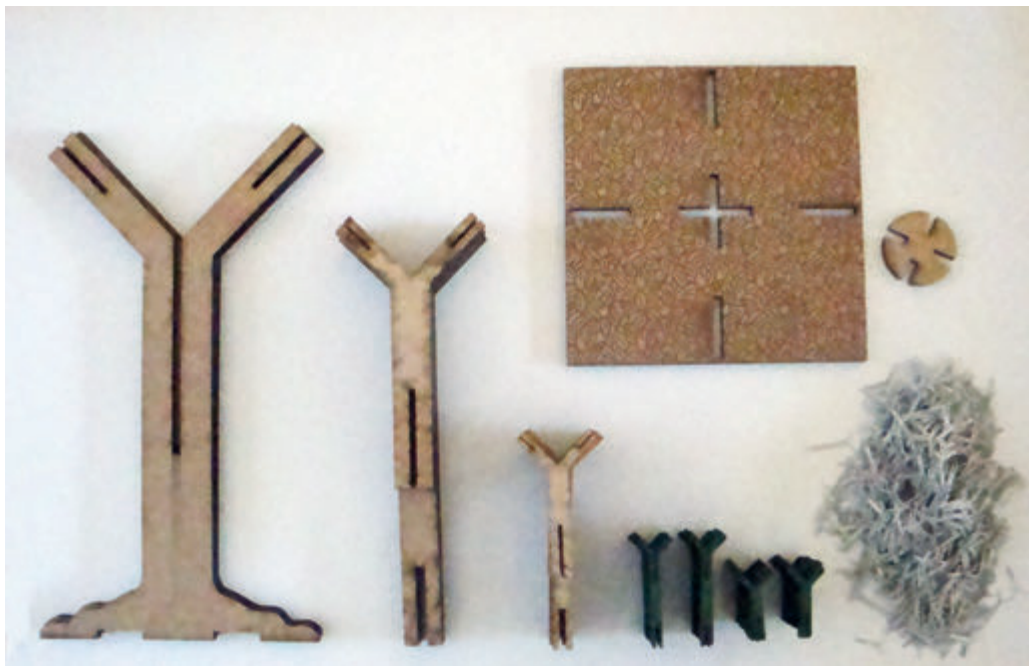


Figure 10: From left to right: the 2 pieces for the trunk of the tree, 8 for the branches, and the 32 twigs; 128 green leaves and 512 white blossoms complete the tree.



Figure 11: The result with 4 generations.



Figure 12: A branch with the 5 generation (patience was lacking to construct the complete tree up to 5 generations).



## ACTIVITY 2: 3D SIERPINSKI IN BALLOONS

### Teacher's guide

**Materials:** Balloons and pumps. These balloons are cheap and the small pumps come with them. The pumps are not strictly necessary, you may need rubber bands to tie off the balloons

**Objective:** Students will become familiar with 3D fractals properties, and experience how slowly or quickly each generation grows.

**Vocabulary:** Fractals, 3D space.

**Activity Sequence:**

1. Blow up two balloons and put them together as shown below. Try to respect a given edge length as well as possible, since this will turn out to be important in the end.

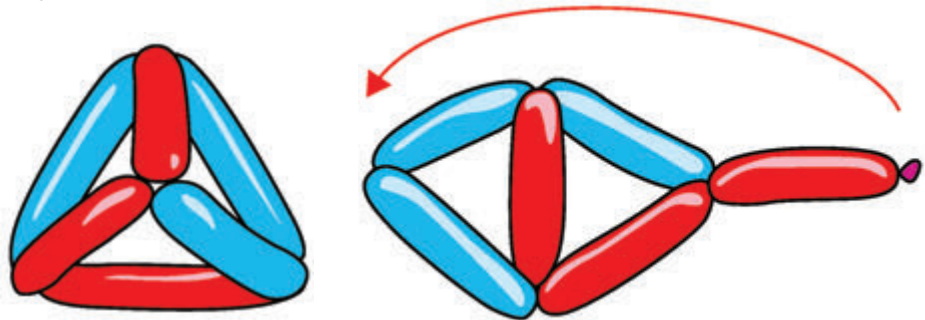


Figure 13: Construction of a tetrahedron using 2 balloons.

2. Make a Sierpinski gasket using 4 balloon tetrahedrons.

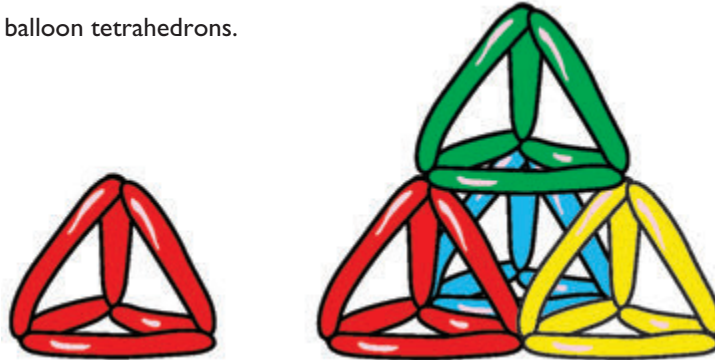


Figure 14: Construction of a Sierpinski gasket using 8 balloons.

3. Make a greater Sierpinski gasket using 4 balloon tetrahedrons and thus using 32 balloons.

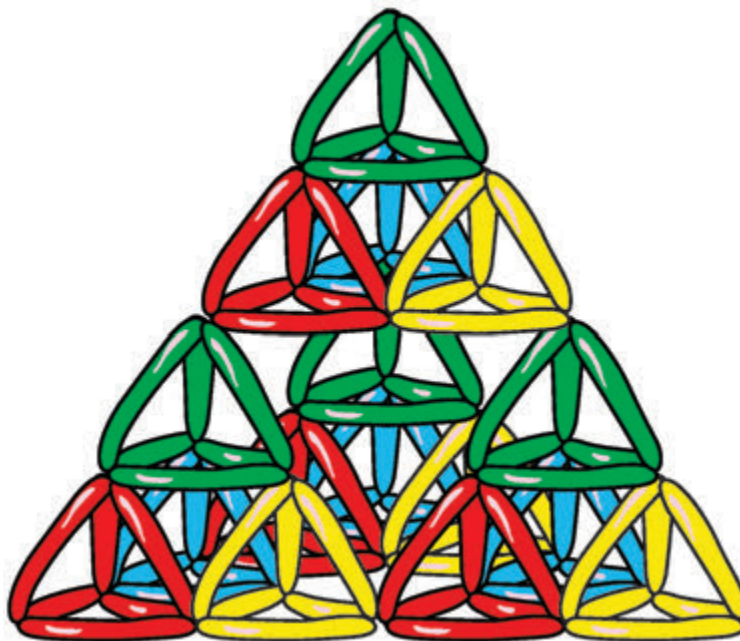


Figure 15: Construction of a Sierpinski gasket using 8 balloons.

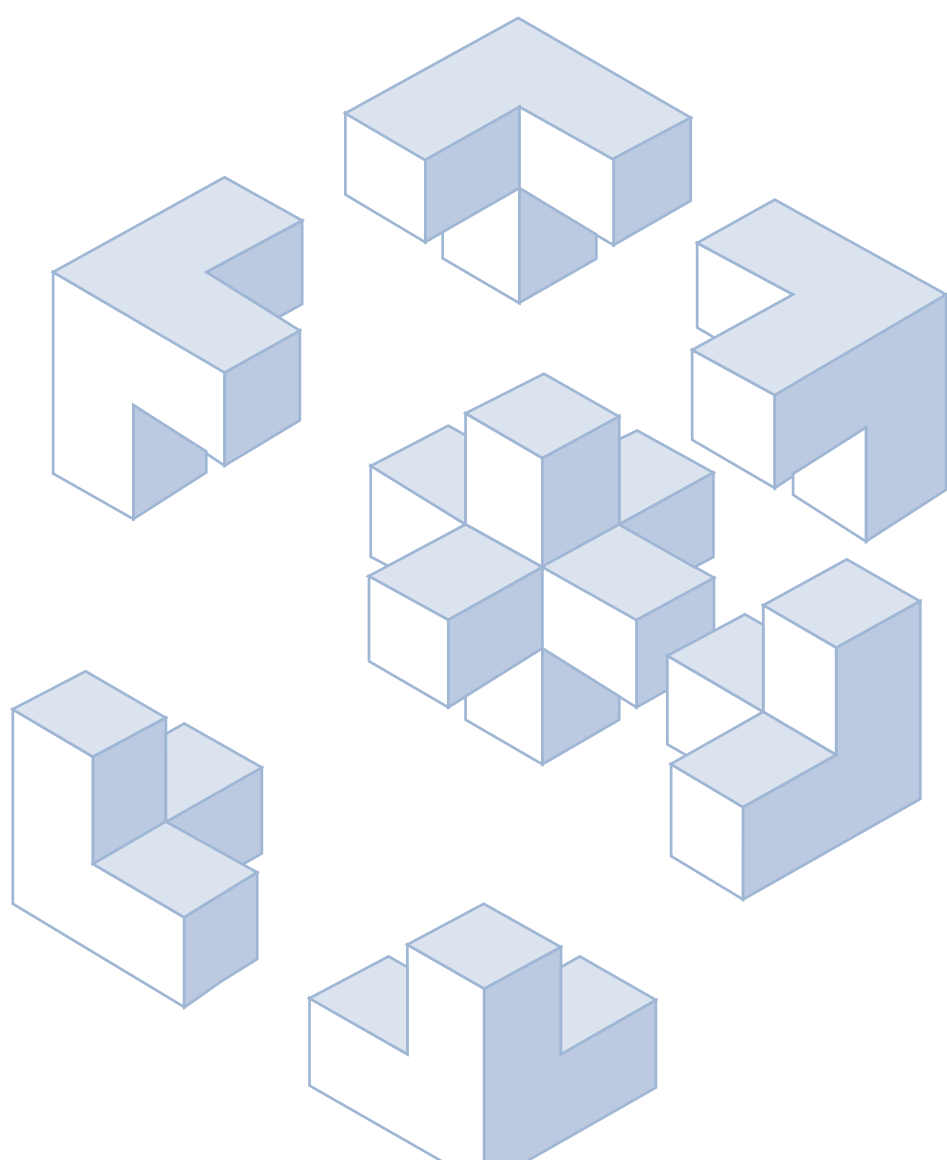
Ambitious mathematical artists or artistic mathematicians can try to break the Guinness world record for the largest balloon Sierpinski gasket. It was set on March 16th 2014, using 2,048 balloons, in 1,024 pyramids, measuring just over 7m tall and taking six hours to put together.



Figure 16: Belgrade Metropolitan University student Jovana Kovac in a Sierpinski-like balloon gasket made by her together with her colleague, photographer Jovana Petrovic.

## References:

- [1] D. Avnir, O. Biham, D. Lidar and O. Malcai, "Is the Geometry of Nature Fractal?", *Science*, 279 (5347), pp. 39-40 (1998).
- [2] P.M. Bliss and D.A. Brown, "Geometric properties of three-dimensional fractal trees", *Chaos, Solitons and Fractals*, 42, Elsevier, pp. 119–124 (2009).
- [3] E. D. Demaine, M. L. Demaine and V. Hart, "Balloon Polyhedra", in *Shaping Space: A Polyhedral Approach*, edited by Marjorie Senechal, Second Edition, pp. 33-40 (2013).
- [4] R. Eglash, "African Fractals. Modern Computing and Indigenous Design", Rutgers University Press, New Brunswick NJ (1999) .
- [5] R.M. Frongillo, E. Lock, and D.A. Brown, "Symmetric fractal trees in three dimensions", *Chaos, Solitons and Fractals*, Elsevier, 32, pp. 284–95 (2007).
- [6] B. E. Pons, "Generalized self-contacting symmetric fractal trees", *Symmetry: Culture and Science*, Vol. 21, Nos. 1-4, pp. 333-351 (2013).
- [7] T. Verhoeff and K. Verhoeff, "Mitered Fractal Trees: Constructions and Properties", *Proceedings of the conference "Bridges: Mathematics, Music, Art, Architecture, Culture"*, pp. 25-32 (2011).







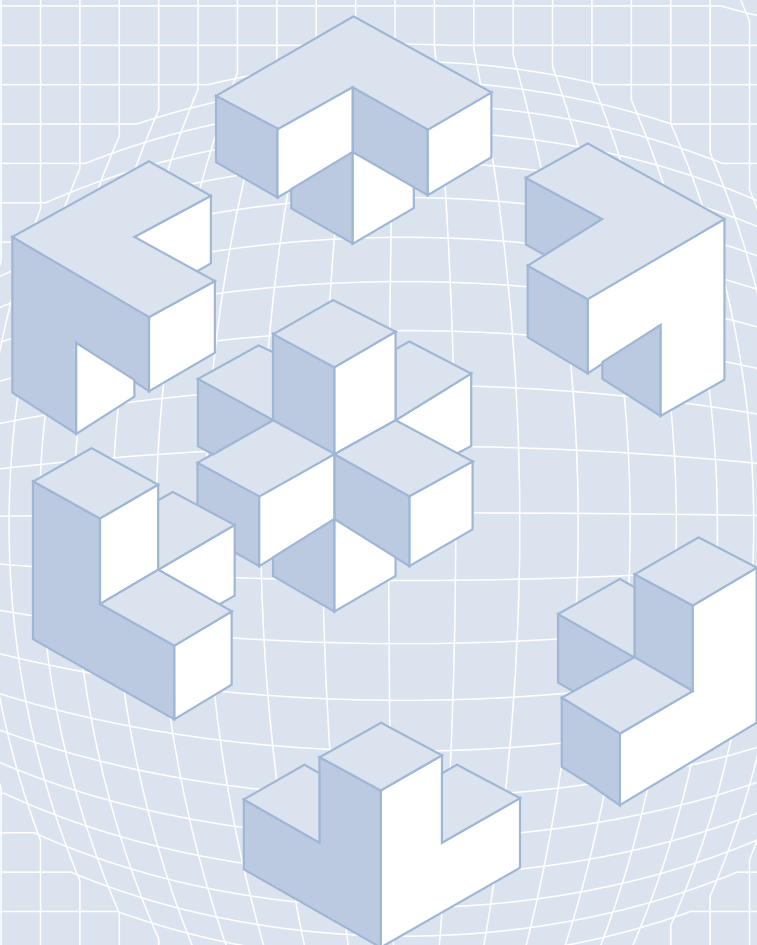
VISUALITY &  
MATHEMATICS

EXPERIENTIAL EDUCATION  
OF MATHEMATICS THROUGH  
VISUAL ARTS, SCIENCES  
AND PLAYFUL ACTIVITIES

GEORGE HART

"SLIDE-TOGETHER"  
GEOMETRIC PAPER  
CONSTRUCTIONS

PAPER SCULPTURES

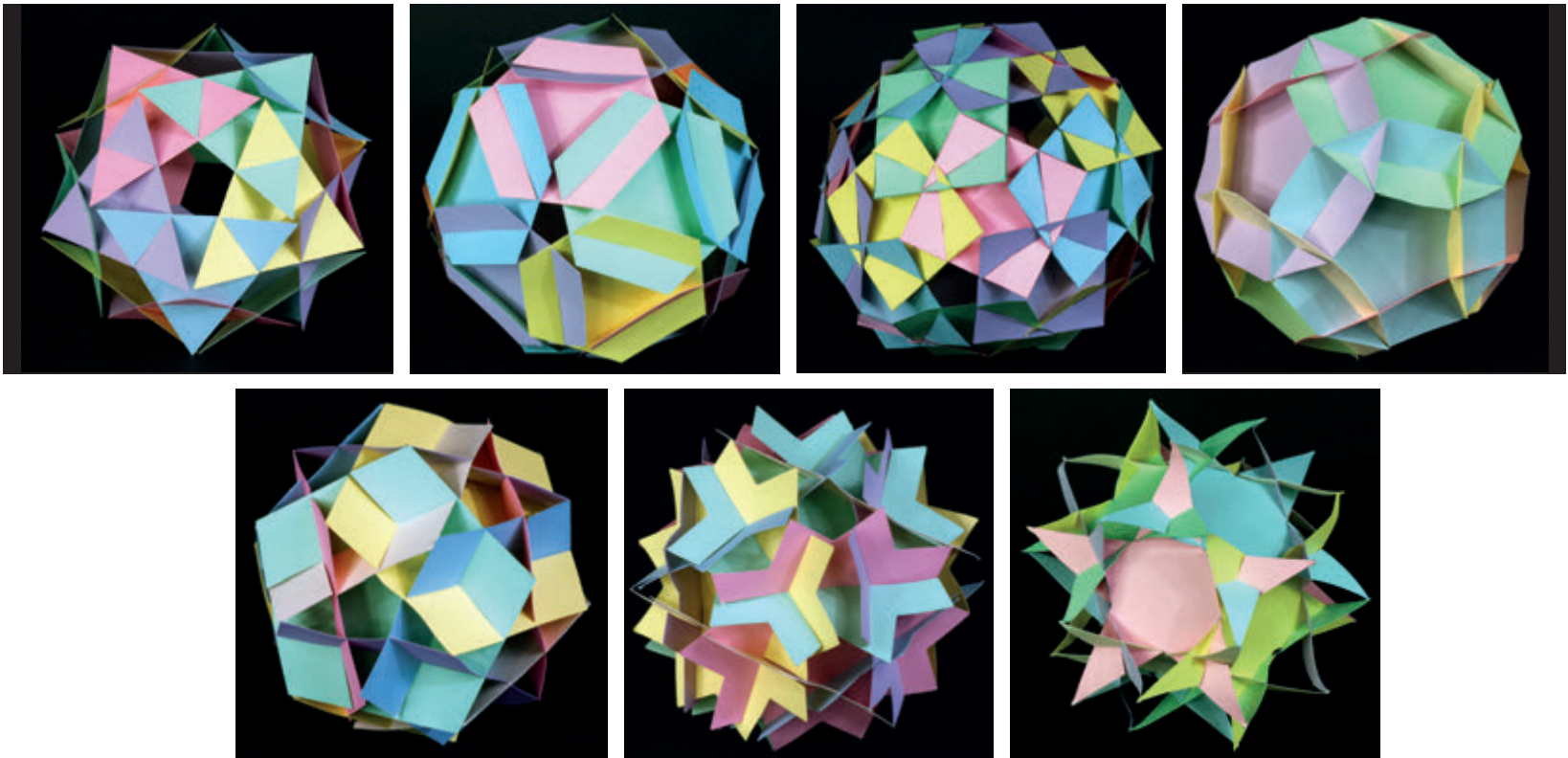


**GEORGE HART** is a sculptor and an inter-departmental research professor at Stony Brook University. He holds a B.S. in Mathematics and a Ph.D. in Electrical Engineering and Computer Science from MIT. Hart is a co-organizer of the annual Bridges Conference on mathematics and art and the editor for sculpture for the Journal of Mathematics and the Arts. His artwork has been exhibited widely around the world. Hart co-founded the Museum of Mathematics in New York City and designed its initial exhibits. He also makes videos that show the fun and creative sides of mathematics. See [georgehart.com](http://georgehart.com) for examples of his work.



**Introduction.** This activity consists of seven attractive constructions which are fun and relatively inexpensive to make because one simply cuts out paper pieces and slides them together. A number of mathematical skills are developed, concerning geometric structure, coloring patterns, and concrete and mental visualization. I have found these to be good classroom activities for middle-school, high school, and college students. Furthermore, as team-building projects, these work well if assembled in groups of two or three students. That encourages collaboration and mathematical communication.

Each “slide-together” is made from identical copies of a single type of regular polygon (e.g., just squares or just triangles) with slits cut at the proper locations. I make them from colored card stock, simply photocopying the templates onto the sheets. In most cases, glue or tape is not needed if you use a stiff stock. But you might want to use a small dot of glue at the corners or bit of scotch tape on the interior to fasten the components together. Having the corners meet crisply is the key to producing a neat geometric impression.



*Paper “Slide-together” Constructions based on triangles, hexagons, squares, decagons, pentagons, decagrams, and pentagrams*

**Differentiating Instruction.** The seven models are illustrated above in approximately increasing order of construction difficulty. I suggest starting with the triangles or hexagons, which are straightforward to make based on the image. The one with squares is more challenging because there are places where three sheets meet. Instructions for it are detailed below. The later ones, with decagons, pentagons, decagrams, and pentagrams are increasingly more difficult. One strategy is to have everyone in a class make the triangles model and then have different teams each work on a different one of the remaining models. Assign the more difficult ones to the teams which want a greater challenge. Combining the results can make a very attractive display. A mobile I made with all seven models was exhibited first at the 1997 *Math and Art* conference at SUNY Albany, NY and then at the Goudreau Museum of Mathematics, in New Hyde Park, New York.

## CONSTRUCTING THE 30-SQUARES "SLIDE-TOGETHER"

**Copy and Cut.** For one model, use five sheets of "card stock" of five different colors. (Ordinary paper is too thin. Card stock is a heavy weight paper, stiffer than standard paper, but thin enough to snake through the rollers of a copy machine or laser printer. Most copy shops have a selection of colors on hand that they can copy on to for you, or you can buy it by the ream to put in your own copier.) Copy the squares template (below) on to the five sheets. If you wish, you can scale it up a bit to make six 3.5 inch squares exactly fit in one sheet of paper; it is only essential that all thirty squares be the same size. If only a single color of paper is used, the construction still works geometrically, but much of the beauty is lost.

Using scissors, cut on the lines to release thirty squares. Individually cut the four slits in each, i.e., do not stack squares and try to make several slits with one cut as that will be too inaccurate. *Neatness counts!* You do not need to cut all the pieces before beginning assembly. You can start construction once you have cut and slit at least one square of each color.

In what follows, keep in mind the following:

- (1) the squares are planar; you will bend them temporarily during assembly but they should end up flat;
- (2) when two squares are slid completely into each other, two edges of one square intersect two edges of the other (one crossing occurs at each end of the slit); and
- (3) each square will join to four squares of the four other colors, e.g., a blue square never touches another blue square.

**Cycle of Five.** Notice that there are two long slits and two short slits in each square. You will always slide a long and a short slit into each other. Begin by joining two squares of different colors. Then observe in the first photo above that the central five-fold opening is surrounded by five squares and see how two of those five are arranged like the two squares you just joined. Continue the pattern and add a third square, a fourth, and a fifth. Join the fifth to the first to complete a cycle around a five-fold opening. Be sure always to keep the corners of the squares all on the outside of the construction. A common problem is not sliding the slits completely into each other; you can detect this by noticing that the edges do not intersect.

**Three-way Corners.** At this stage, the joints are free to rotate, so the assembly will be very flexible and some joints may disassemble spontaneously. If this happens, just repair the joints to maintain the pentagon opening. What locks the parts together are the "three-way corners" which are added next. To visualize where they go, keep in mind that of each square's four edges, two (opposite) edges will touch pentagonal openings and the remaining two (opposite) edges will touch three-way corners. Observe this in the photo above.

To make a three-way corner between squares A and B, you choose a new square C and join it into both A and B. The first issue is to determine what color C should be. The trick is to look directly across the pentagon from where A and B touch and see what color square is there; choose a square C of that same color. The second issue is to make the three-way corner symmetric with a neat little triangle at its center. The trick to this is to first join C into A and B with a kind of rotation of C, and then temporarily bend and unbend the little points of A, B, and C as needed to get around each other and make a sort of spiral. It is easier to do than explain in text. Typically some students discover this then demonstrate it to their peers.

**Completing the Structure.** Once this trick is mastered it is straightforward to create another three-way corner, and another, etc, so all five initial joints are locked. In each case, the color of the new square that is added must be determined by looking across the pentagonal opening to match the color of the square opposite. When all five of the original joints are locked in this way, you will have used a total of ten squares, so you are a third done. Completing



the structure is just a matter of noticing there are several incomplete pentagonal openings, choosing any one to complete, and locking its joints, etc. until all thirty squares have been used. Double check as you go along that every opening is surrounded by five different colors and each square joins with four other squares of the four other colors. If properly made, the six squares of any color are arranged like an exploded cube.

## CONSTRUCTING THE OTHER SIX "SIDE-TOGETHERS"

Similar techniques are used to assemble the six other slide-togethers. Each can be visualized as sets of intersecting polygons, with the slits being used to allow the planes of the paper to get through each other. One tricky issue is the choice of color of each part so the whole arrangement is symmetric. A second issue is the technique of making more difficult 3-way corners with larger parts that have to bend around each other. The illustrations above are guides. In each case, interesting patterns of edges are formed, often five-pointed stars.

The one with triangles and the one with hexagons each have twenty components and should be made with four parts in each of five colors. These do not have three-way corners, so they are easier to assemble in that respect, but are correspondingly prone to self-dis-assembly with certain types of paper. A bit of tape can be used on the interior to lock the slots together if needed. Also, dots of glue can be used to hold the corners to each other. If properly assembled, the four parts of any color lie in the planes of a regular tetrahedron and all five colors appear around each pentagon opening. The one with triangles is especially interesting because among its edges you can see the edges of five cubes; if at first you do not see them, they may pop out if you rotate the model slowly.

The four remaining models each have twelve components. For each, make two parts in each of six colors and assemble them so pairs of opposite parts are always the same color. Each part will touch five neighbors of the other five colors. The three-way corners can be tricky at first. The most difficult one is the construction of twelve pentagrams, because the segments where two stars pass through each other have two pairs of slits to join instead of just one. It would be impossible to give detailed instructions for these. All I can suggest is that at each step you first visualize where you want a part to be, then arrange for it to be there.

## CLASSROOM IDEAS FROM MIDDLE GRADES TO ARCHITECTURAL DESIGN

Before attempting these constructions, students should have built all five Platonic solids, to be familiar with their symmetries. The completed slide-together models can be related to the regular polyhedra and used to explore ideas of counting or symmetry. For example with the 30 squares construction, you can ask: How many "three-way corners" are there? (Answer: 20, they correspond to the 20 faces of a regular icosahedron. One way to count them is based on the fact that each of 30 squares touch two three-way corners, and it takes three such contacts to make each, so  $30 * 2 / 3$  gives 20.) How many 5-sided openings are there? (Answer: 12, corresponding to the 12 faces of a regular dodecahedron, calculated as  $30 * 2 / 5$ .) How many 5-fold rotation axes are there? (Answer: 6. One connects the centers of each pair of opposite 5-fold openings.)

One possible advanced project is to have students make their own templates using either straightedge and compass or a computer drawing program. The key in many cases is to start with a regular polygon and find points which divide the edges in the golden ratio. (This follows from the golden-ratio properties of a five-pointed star.) The cuts where parts slide into each other should add up to the length of the segment of intersection.

After practice with these melon-sized models, the idea can be applied at a larger scale. Large cardboard versions about five feet in diameter have been made by students in a college-level architectural design course taught by Prof. Patricia Muñoz at the University of Buenos Aires.

At the high school or higher level, one can use the constructed models to explore topics in combinatorics, e.g., in the 30 squares: How many different cycles of five colors are possible around a five-sided opening? (Answer: 24, which is  $5!/5$  because there are  $5!$  permutations of the colors, then "equate" groups of five that are cyclic rotations.) How many different cycles are present in one model? (Answer: 12, one around each of the 12 openings.) So how many differently colored models are in the classroom? (Ans: 2—If the order of initial cycle of five colors is chosen randomly, roughly half the class will have one coloring pattern and half will have the other.) What determines which 12 of the 24 possible cyclic orders are found in the same model? (Answer: The "even" permutations of the five colors are in the same model.)

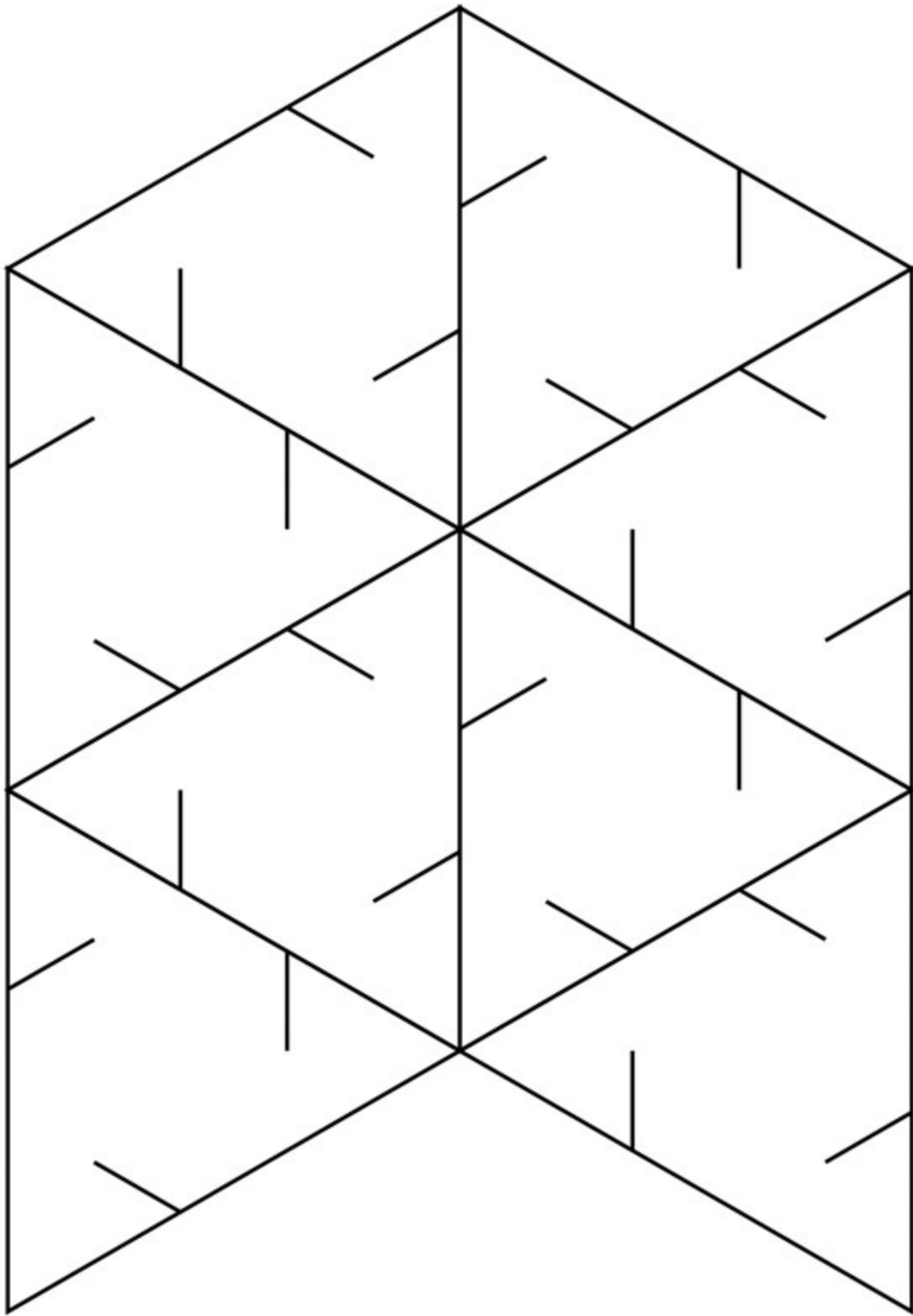
## REFERENCES

Charles Butler described to me the design of the one with triangles and the one with squares. I designed the others based on an assortment of uniform polyhedra.

A YouTube video "Seven Slide-Together Constructions" illustrating these models is available at **<http://youtu.be/Yq0La4rEarc>**

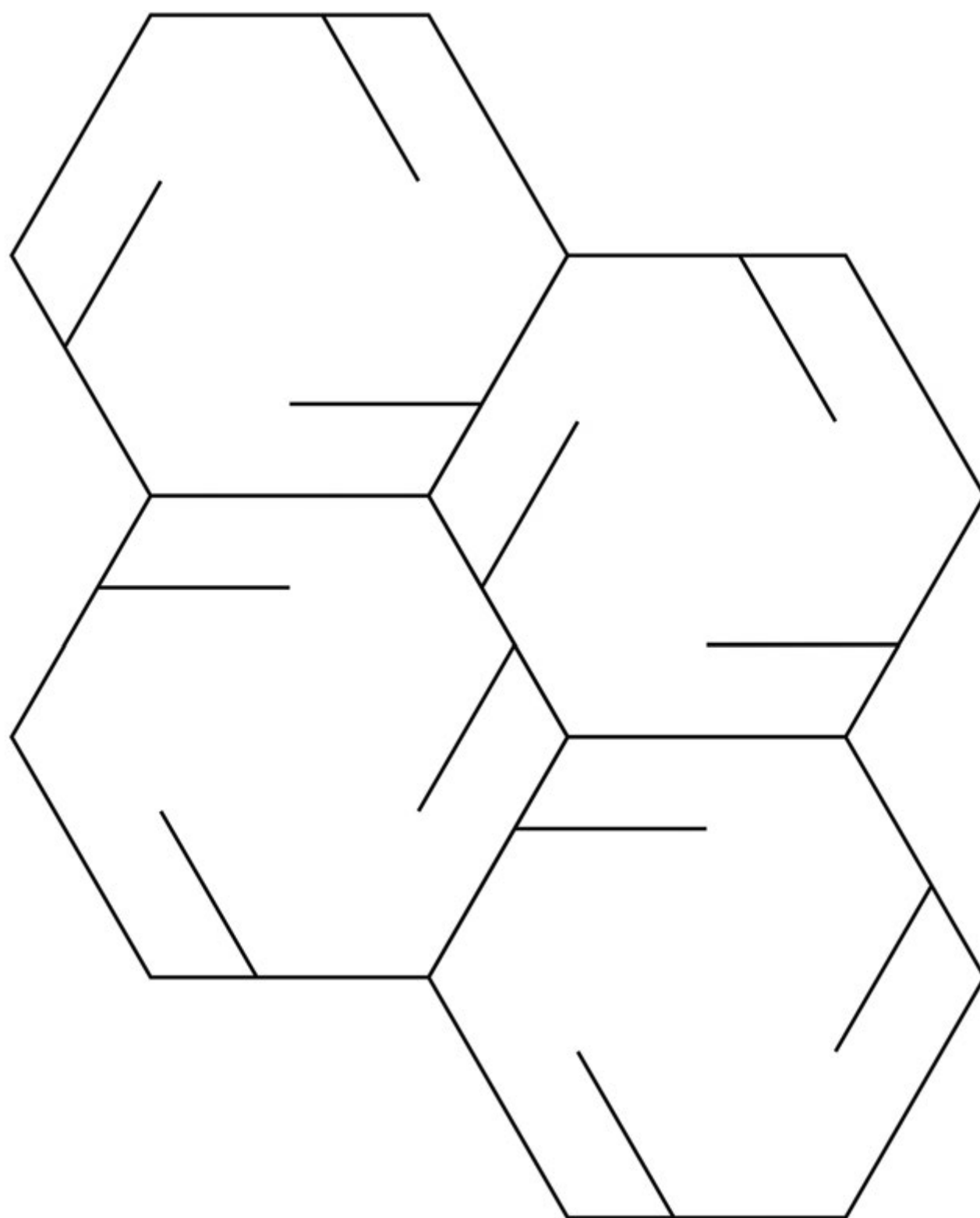
**<http://www.georgehart.com>** includes 3D "virtual reality" models of all seven, which rotate in three dimensions in one's web browser with an appropriate "plug-in" viewer. This gives a richer sense of the structure than 2D images, so may be a better guide for assembly.

Portions of this material appeared in the 2004 Bridges Conference booklet of Teacher Workshop materials.

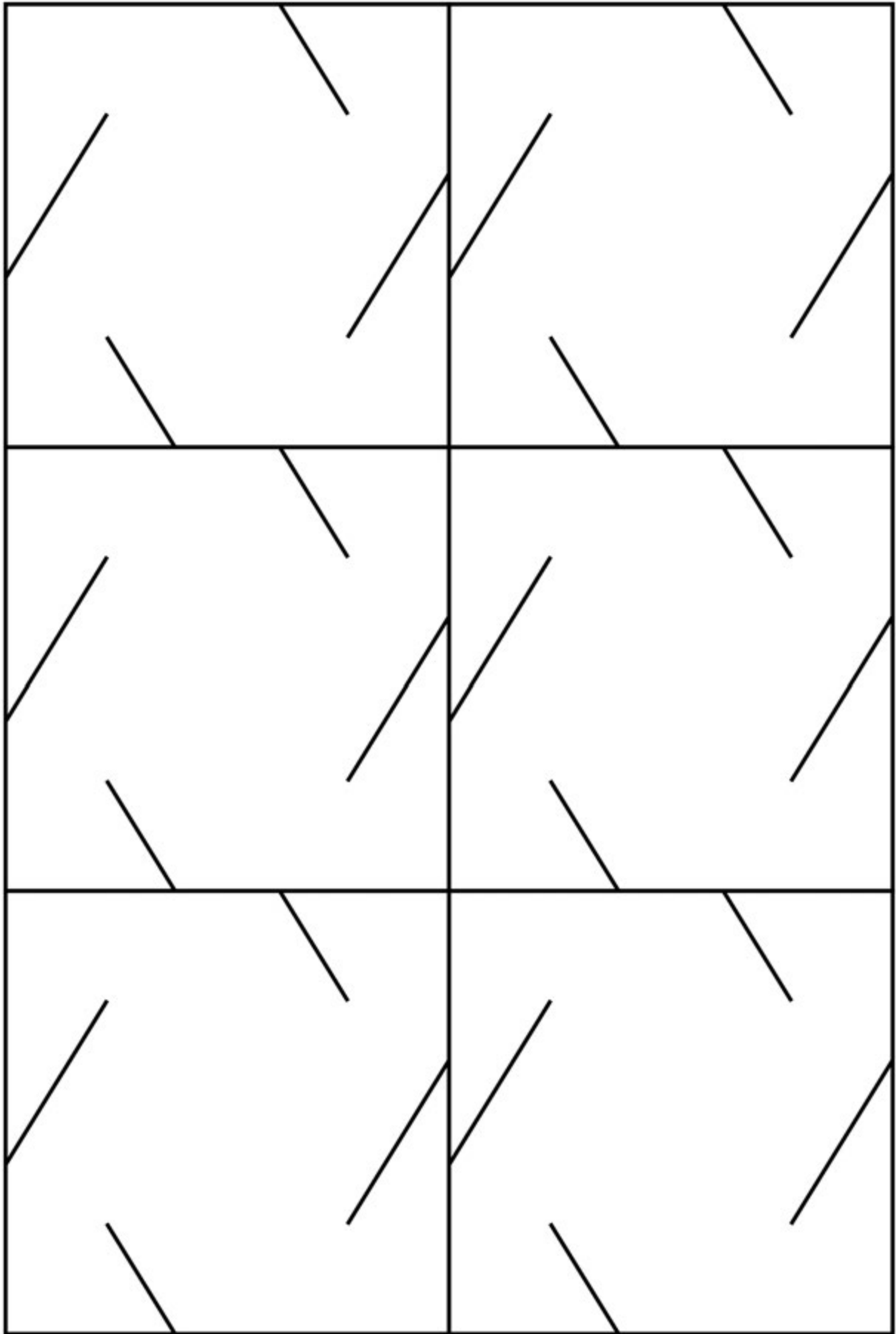


*Triangle Slide-together Template — make five copies for two models*

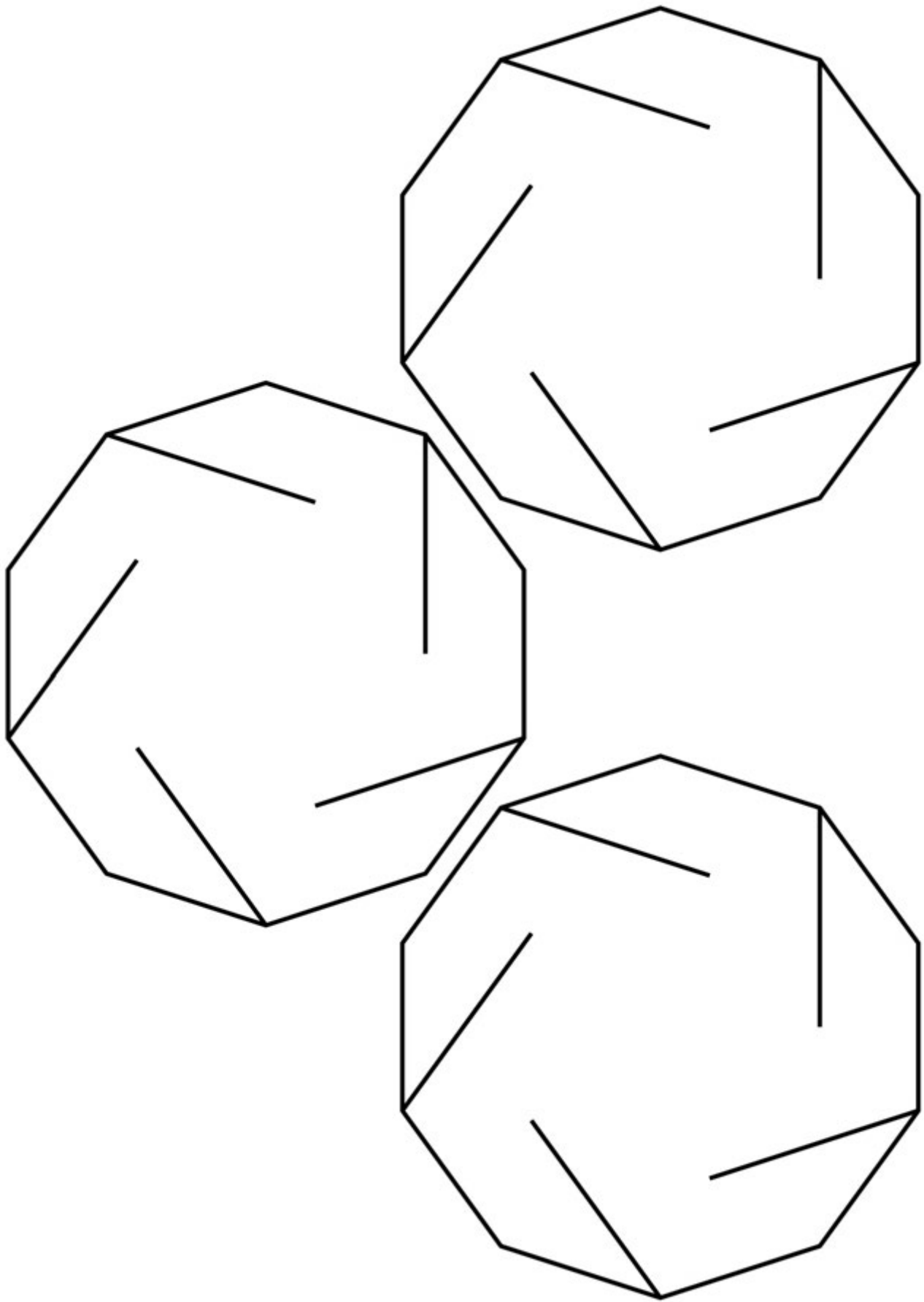




*Hexagon Slide-together Template — make five copies for one model*

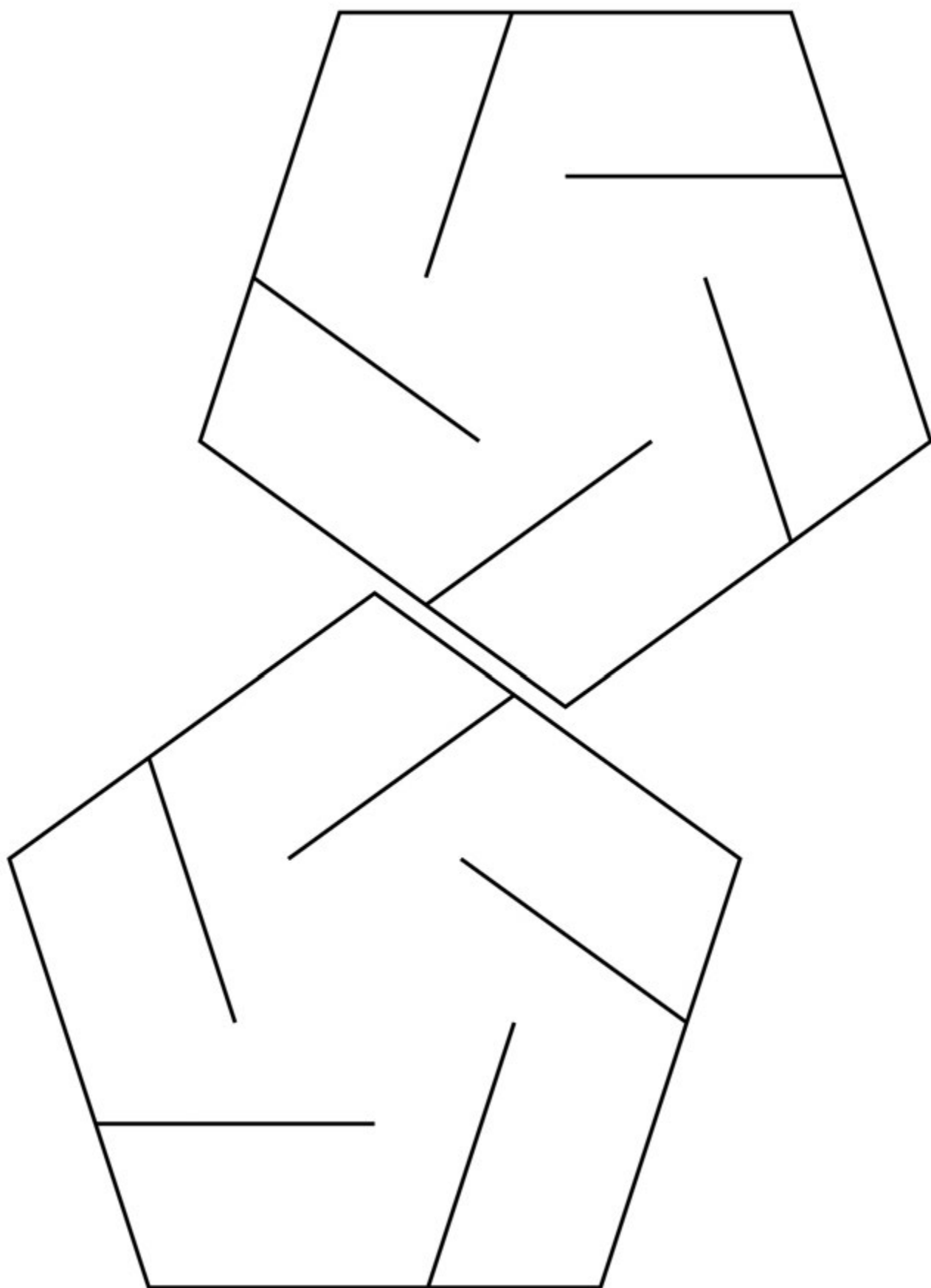


*Square Slide-together Template — make five copies for one model*

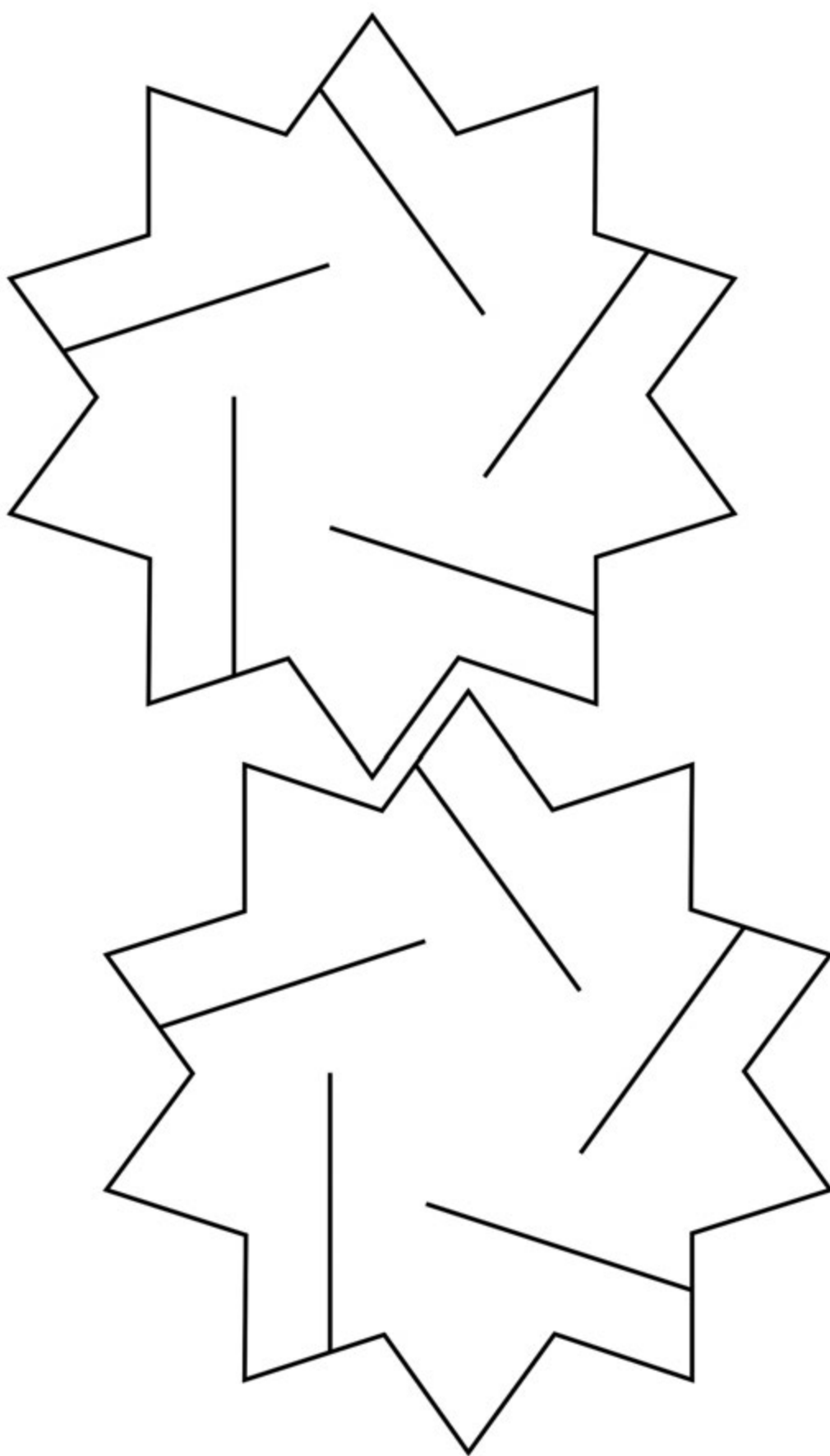


*Decagon Slide-together Template — make six copies for 1.5 model*

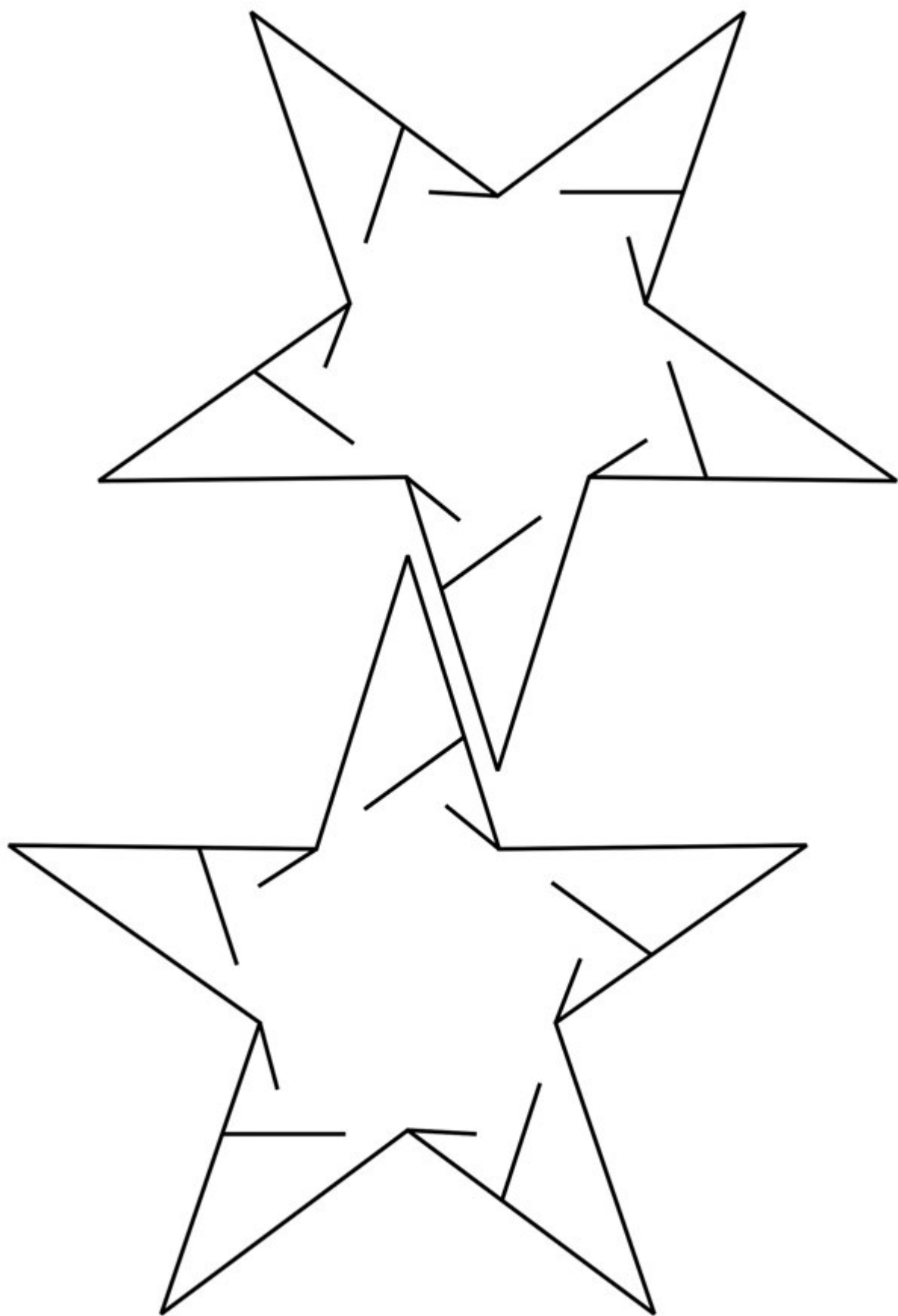




*Pentagon Slide-together Template — make six copies for one model*



*Decagram Slide-together Template — make six copies for one model*



*Pentagram Slide-together Template — make six copies for one model*





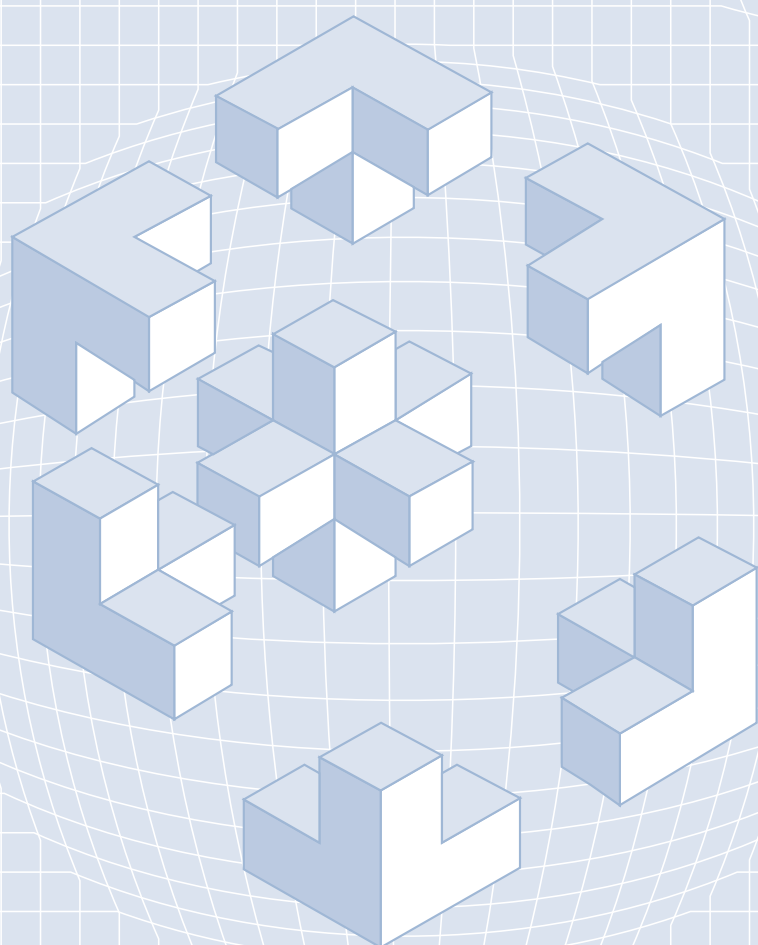
VISUALITY &  
MATHEMATICS

EXPERIENTIAL EDUCATION  
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RINUS ROELOFS

# LEONARDO'S ELEVATED POLYHEDRA MODELS

PAPER SCULPTURES



**RINUS ROELOFS** was born in 1954. After studying Applied Mathematics at the Technical University of Enschede, he took a degree from the Enschede Art Academy with a specialization in sculpture. His commissions come largely from municipalities, institutions and companies in the Netherlands, but his work has been exhibited further afield, including in Rome as part of the Escher Centennial celebrations in 1998. Since 2003 he has been a presenter at the annual Bridges conference, a conference on connections between art and mathematics.



## PUBLICATIONS

*Ecken und Kanten – Europaweit, 2014, catalogue exhibition Mathematikum Giessen, Germany.*

*The Discovery of a New Series of Uniform Polyhedra, 2013, Bridges Proceedings.*

*Connected Holes, 2008, catalogue exhibition Technical University of Enschede, Netherlands.*

## INTRODUCTION

The main subject of my art is my fascination with mathematics. And to be more precise: my fascination with mathematical structures. Mathematical structures can be found all around us. We can see them everywhere in our daily lives. The use of these structures as visual decoration is so common that we don't even see this as mathematics. But studying the properties of these structures and especially the relation between the different structures can bring up questions. Questions that can be the start of interesting artistic explorations.

Artistic explorations of this kind mostly lead to intriguing designs of sculptural objects, which are then made from all kinds of materials, like paper, wood, metal, acrylic, etc. It all starts with amazement, trying to understand what you see. Solving those questions often leads to new ideas, new designs.

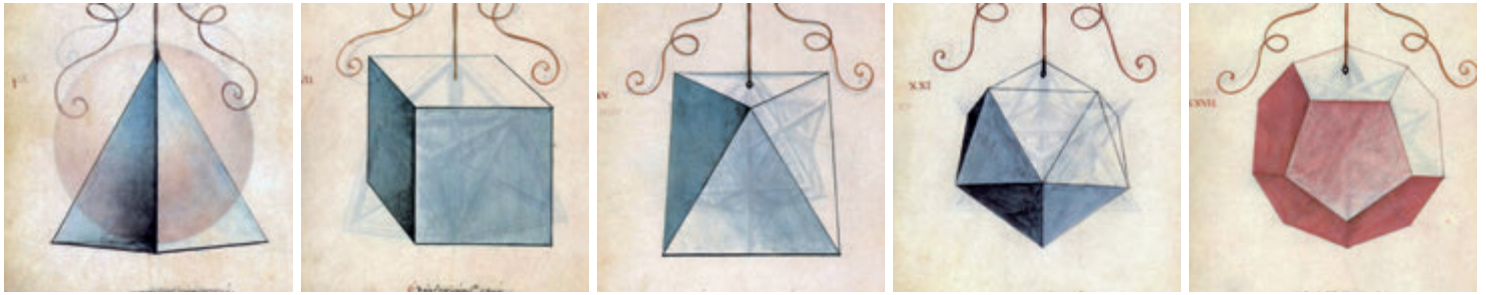
Since I use the computer as my main sketchbook, these ideas come to reality first as a picture on the screen. From there I can decide what the next step towards physical realization will be. A rendered picture, an animation or a 3D physical model made by the use of CNC-milling, laser cutting or rapid prototyping. Most of the time the first physical model is a paper model, simply cut out, folded and glued together.

However many different techniques can be used nowadays, as well as many different materials. But it is all based on my fascination with mathematical structures.

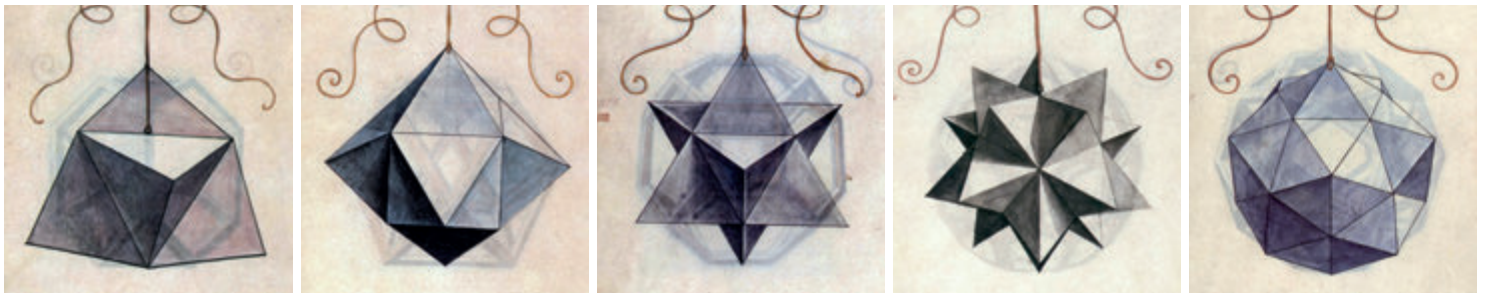
In mathematics the field of Polyhedra deserves special attention. It is nice to build real physical models from which you can learn a lot about the beauty of symmetry and structure. Once you have the first models of the Platonic solids you are inspired and motivated to come up with ideas about variations on these models. In the book "La Divina Proportione" by Luca Pacioli and Leonardo da Vinci you will find drawings of the basic polyhedra and also an interesting variation, called elevation.

Paper models can be made quite easily. For the models described here only three different templates are required, they can be found in the appendix.

In their book “La Divina Proportione” [2] Luca Pacioli and Leonardo da Vinci introduced the elevated versions of all of the Platonic polyhedra as well as of some of the Archimedean polyhedra. The Platonic solids as well as their elevations as they are drawn by Leonardo da Vinci are shown in Figure 1 and 2.



**Figure 1:** Leonardo's drawings of the Platonic solids.

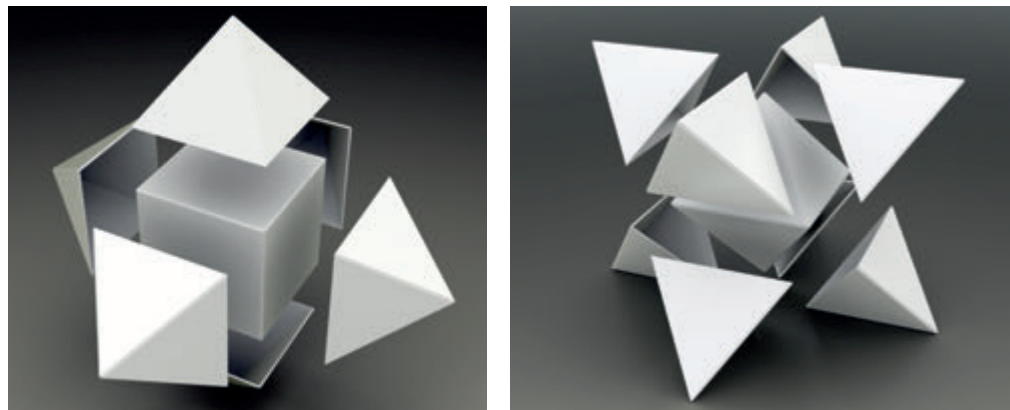


**Figure 2:** Leonardo's drawings of the elevations of the Platonic solids.

What exactly is an elevated version of a polyhedron? In “La Divina Proportione” [3], chapter XLIX, paragraph XI.XII, Pacioli describes the elevated version of the cube as follows: “... it is enclosed by 24 triangular faces. This polyhedron is built out of 6 four-sided pyramids, together building the outside of the object as you can see it with your eyes. And there is also a cube inside, on which the pyramids are placed. But this cube can only be seen by imagination, because it is covered by the pyramids. The 6 square faces are the bottom faces of the 6 pyramids.”

So, in total this object is composed of 24 equilateral triangular faces plus 6 hidden square faces, as can be seen in the exploded views in Figure 3.

**Figure 3:** Exploded views of the elevated cube and octahedron.

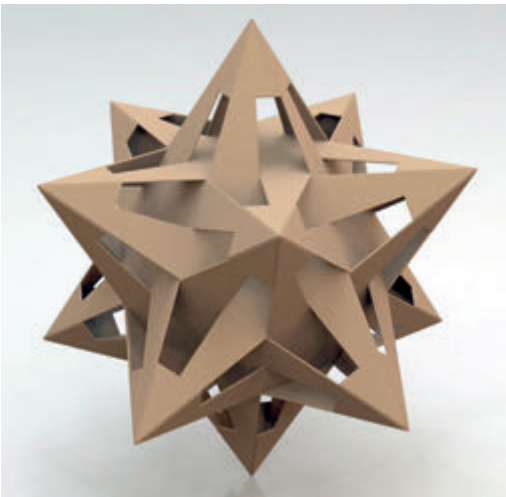


About the “Octocedron Elevatus”, Pacioli writes (Chapter L, paragraph XIX.XX): “And this object is built with 8 three-sided pyramids, that can be seen with your eyes, and an octahedron inside, which you can only see by imagination.”. This means that the object is composed of 32 equilateral triangular faces of which 8 are hidden.

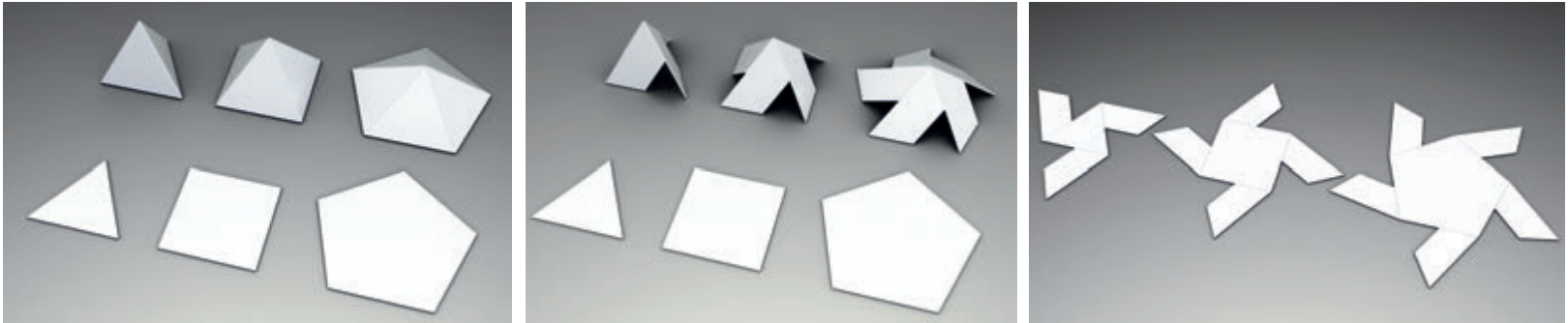
Pacioli describes the process of elevation as putting pyramids, built with equilateral triangles, on each of the faces of the polyhedra. The result of this operation is a double layered object which has many similarities with the stellated version of a polyhedron, a beautiful example of which can be seen in M.C. Escher's print “Gravity”. The way Escher opened up the polyhedron shows us both layers very well. Or in Escher's own words: “This star-dodecahedron is built with twelve five pointed stars. On each of these platforms lives a monster without a tale and his body is captured under a five sided pyramid.”. Escher is talking about pyramids placed on platforms, just like Pacioli's elevations.



**Figure 4:** Find M.C. Escher’s – “Gravity” on the internet. Study the artwork! The work’s basic shape is the Stellated dodecahedron.

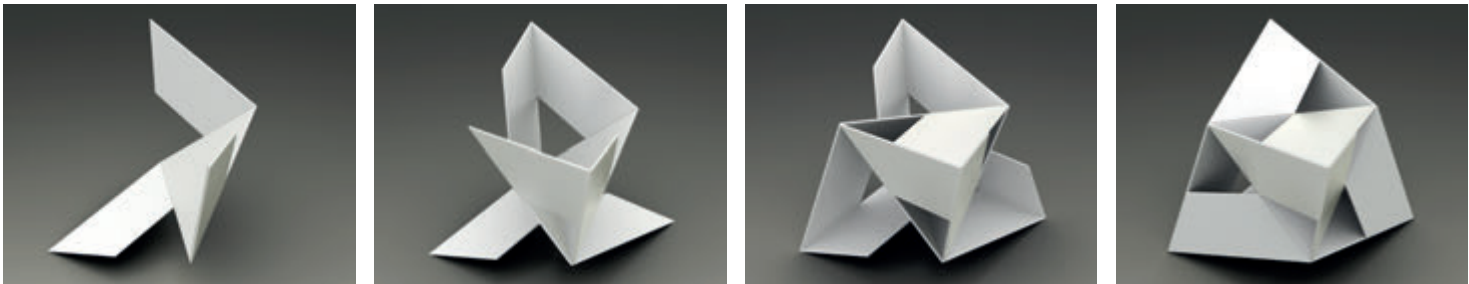


The way Escher opened up the polyhedron turned out to be the perfect solution for making models of the elevated polyhedra.



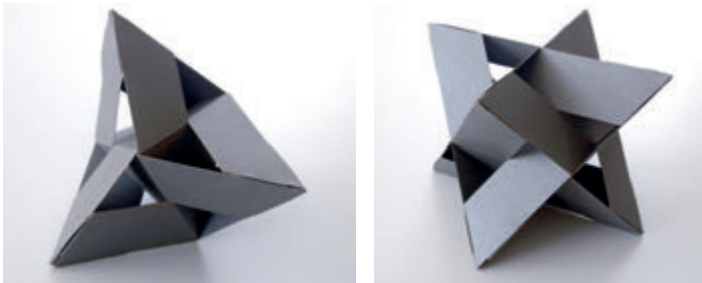
**Figure 5:** Developing of the basic elements for the elevation models.

We can construct the elements that we need to build the model of an elevated polyhedron as follows: connect the faces of the “opened” pyramid to the edges of its base. For the elevated Platonic solids we need to do this with the three, four and five sided pyramids. So we end up with three different building elements (Figure 5, right) of which you can find the drawings in the appendix.



**Figure 6:** Building the elevated tetrahedron.

We start with four triangular elements to first build the model of the elevated tetrahedron. The building process can be described as follows: above each triangular face we have to build a triangular pyramid.



**Figure 7:** Models of the elevated tetrahedron and elevated octahedron.

And with eight of those elements we can build the model of the elevated octahedron.



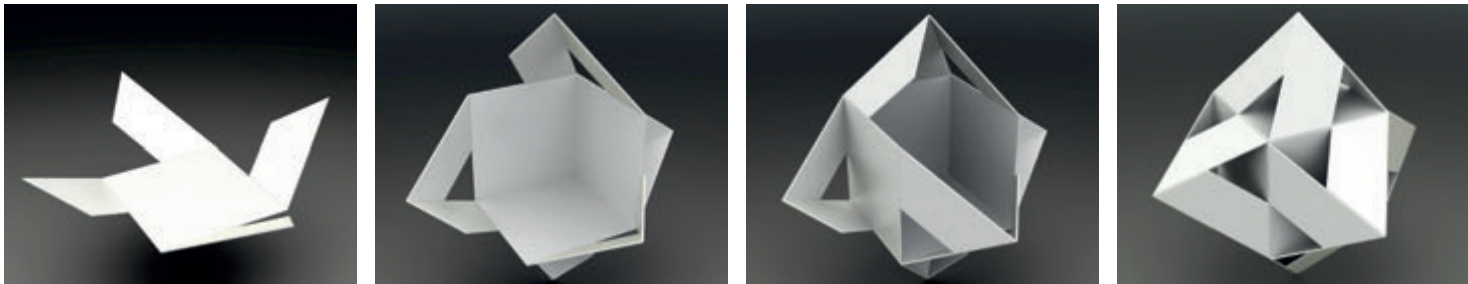
**Figure 8:** *Building the elevated icosahedron.*

For the third model, the elevated icosahedron, we need to cut out twenty triangular elements.

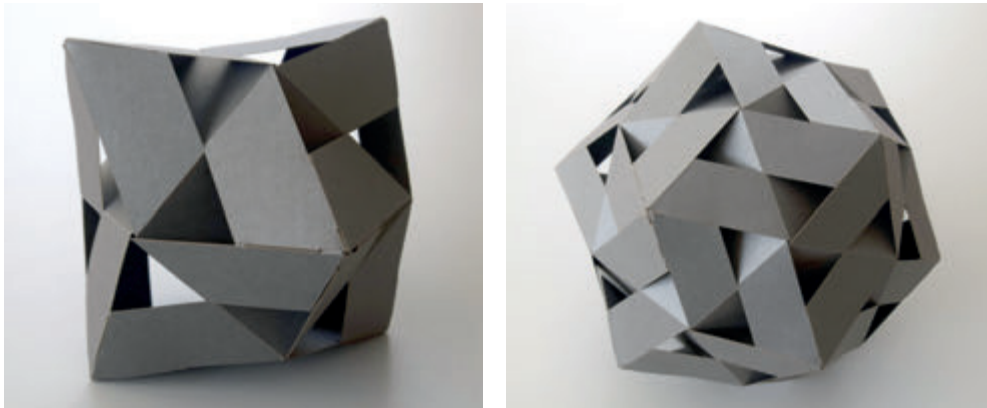


**Figure 9:** *Models of the elevated polyhedra with the triangular element.*

So now we have all the stellated Platonic solids which can be built with the triangular element.  
For the next model, the elevated cube, we need to cut out six square elements. And now we make square pyramids above each of the square faces of the cube.

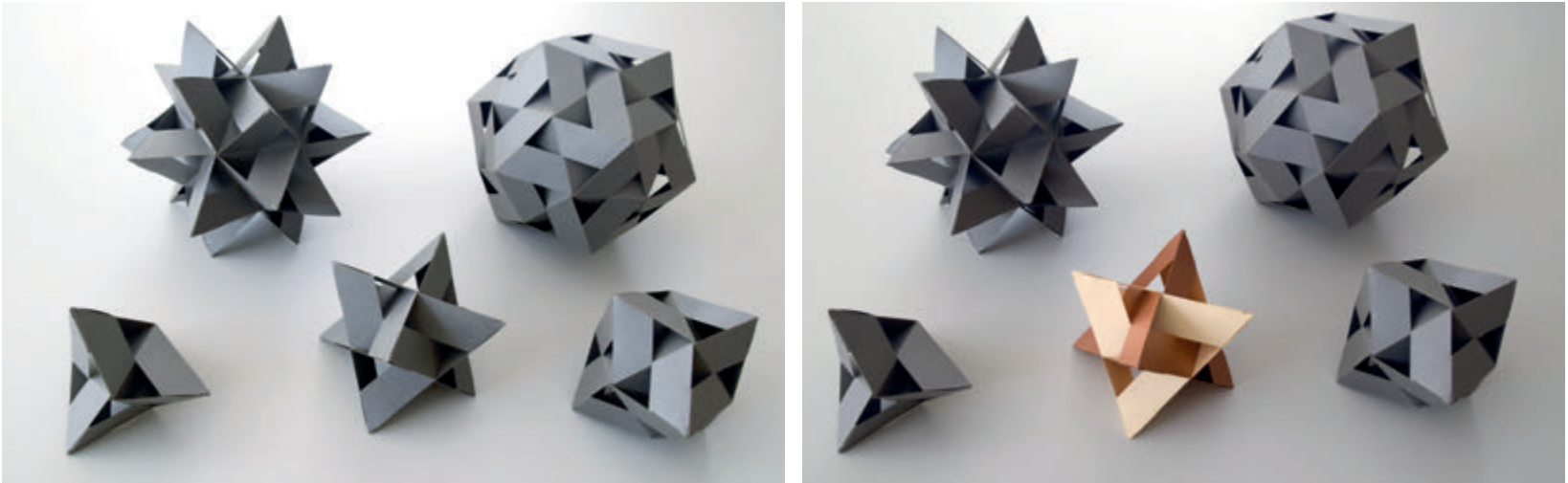


**Figure 10:** *Building the elevated cube*



**Figure 11:** *Models of the elevated cube and elevated dodecahedron.*

The final model of the set of stellated platonic solids, the stellated dodecahedron, can be made with twelve pentagonal elements.



**Figure 12:** *Models of the elevated Platonic solids.*

Looking at the complete set of models we have built so far, we see that one of those models, the stellated octahedron, can be seen as a compound of two tetrahedra. Using two different colors makes this visible and there is a nice resemblance between the paper model and Escher’s “Double Planetoid”. Find M.C. Escher’s “Double Planetoid” on the internet. Study the artwork!



**Figure 13:** *The elevated octahedron is a compound of two tetrahedra.*

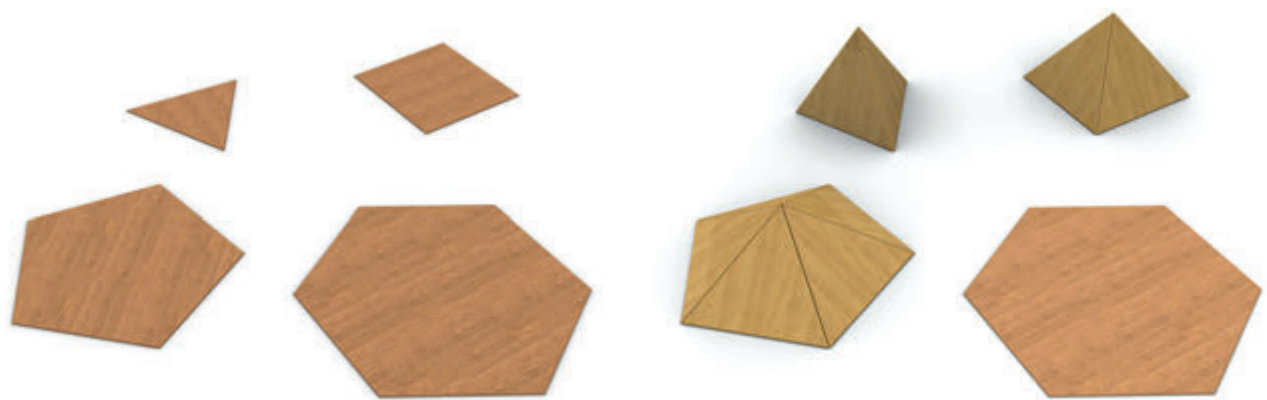
The next step is to look for the possible stellated polyhedra we can get from the Archimedean solids.



**Figure 14:** *The Archimedean and Platonic solids*

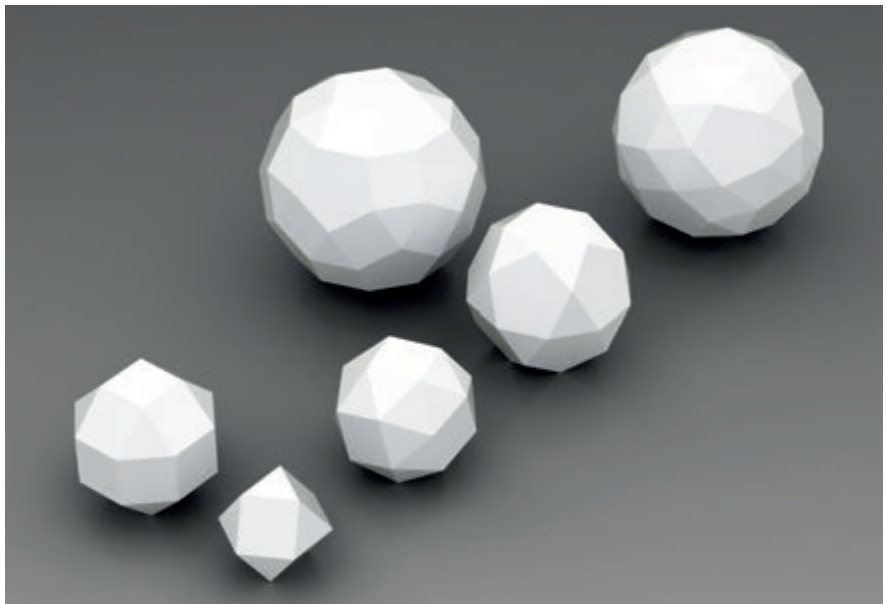


To make a stellation of a polyhedron we have to place a pyramid with equilateral triangular faces on each of the faces of the polyhedron. As you can see in Figure 15, only three, four and five sided pyramids are possible.



**Figure 15:** Possible elevations of regular polygons.

Of the total set of the Archimedean solids, only six solids can be used for the stellation process. They are shown in Figure 16.



**Figure 16:** The six Archimedean Solids which can be elevated.

In the book “La Divina Proportione” we find only three: the cuboctahedron, the icosidodecahedron and the rhombicuboctahedron.



**Figure 17:** Leonardo's drawings of three Archimedean solids.



**Figure 18:** Leonardo's drawings of the elevations of three Archimedean solids.

We can make the paper models of all of them with the three different elements.



**Figure 19:** Paper models of three of the Archimedean solids.

And besides these three Archimedean elevations we can also build the stellated snubcube, the elevated snubdodecahedron and the elevated rhombicosidodecahedron. In total we now have eleven elevated polyhedra. There is a special subset of these eleven polyhedra with interesting properties. In Figure 20 the so called Ring Polyhedra are shown. These polyhedra can be colored with only two colors in such a way that no two adjacent faces have the same color. The elevated versions of these polyhedra will therefore be compounds and we can make the models in two different colors.



**Figure 20:** The five Ring Polyhedra.

In Figure 21 the compounds of Leonardo's elevations are shown.



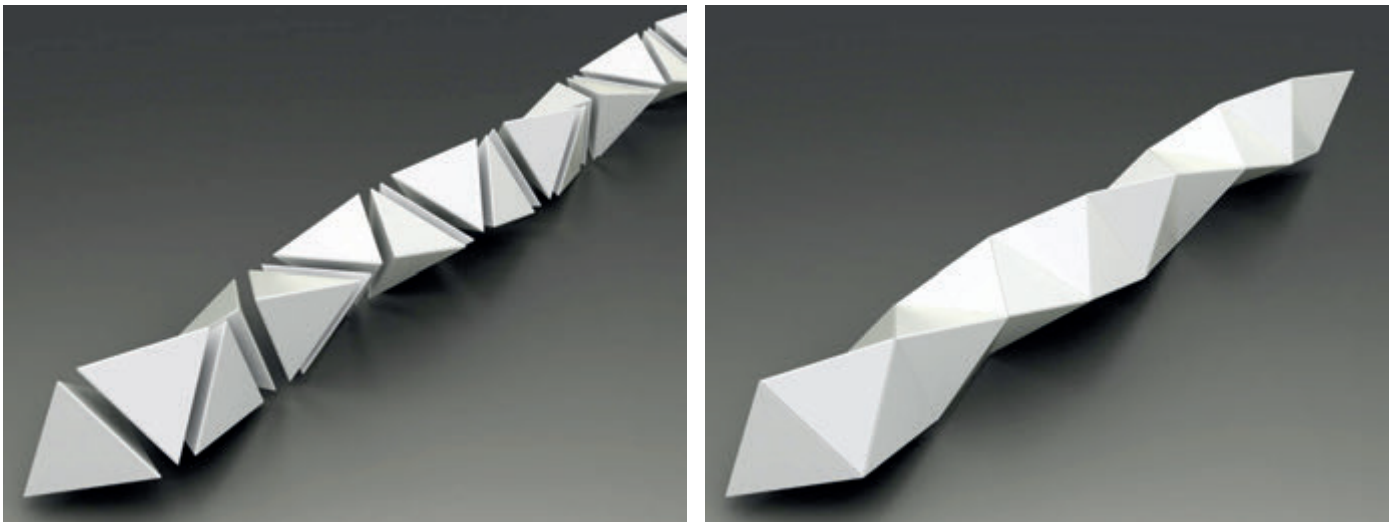
**Figure 21:** *Four of Leonardo's elevated polyhedra are compounds.*

There are many more polyhedra we can build with our elements. The first group we can study is the group of the deltahedra. There are eight convex deltahedra and with our elements we can build elevated versions of each of them.



**Figure 22:** *The eight deltahedra.*

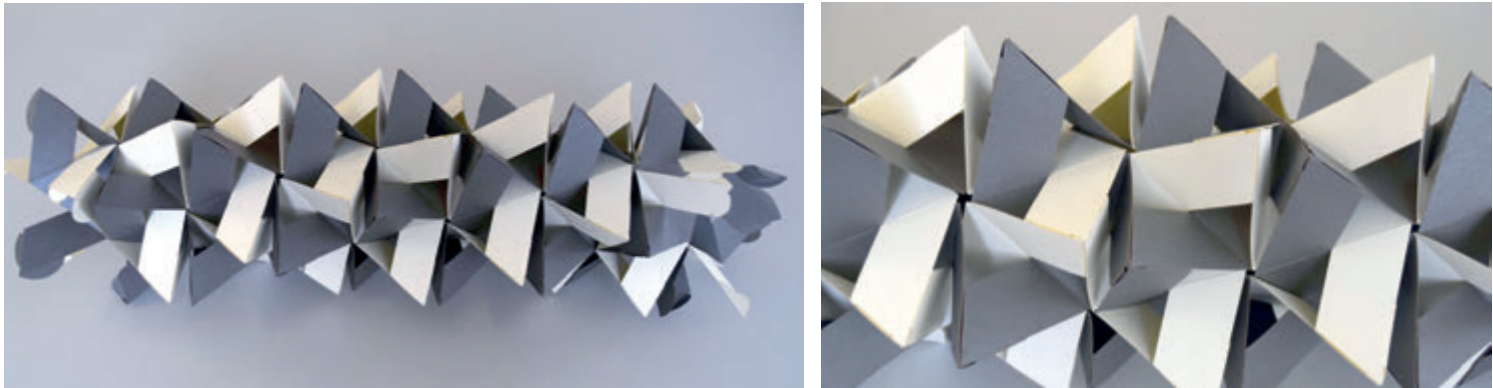
In fact from any polyhedron built with triangles, squares and/or pentagons a paper model of the elevated version can be built with our elements. A nice example is the Tetrahelix.



**Figure 23:** *Tetrahelix*

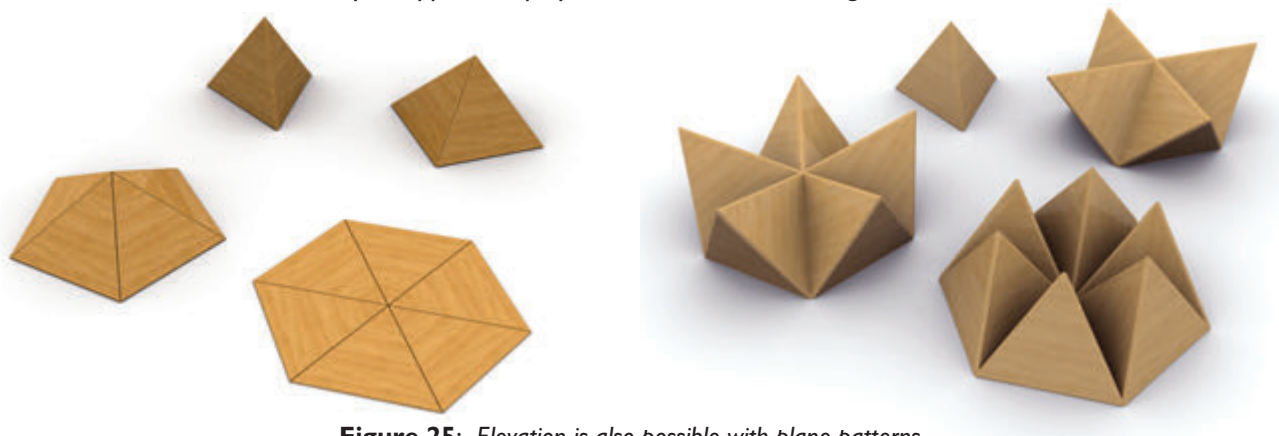


Especially when you use more than one color this will give you nice models.



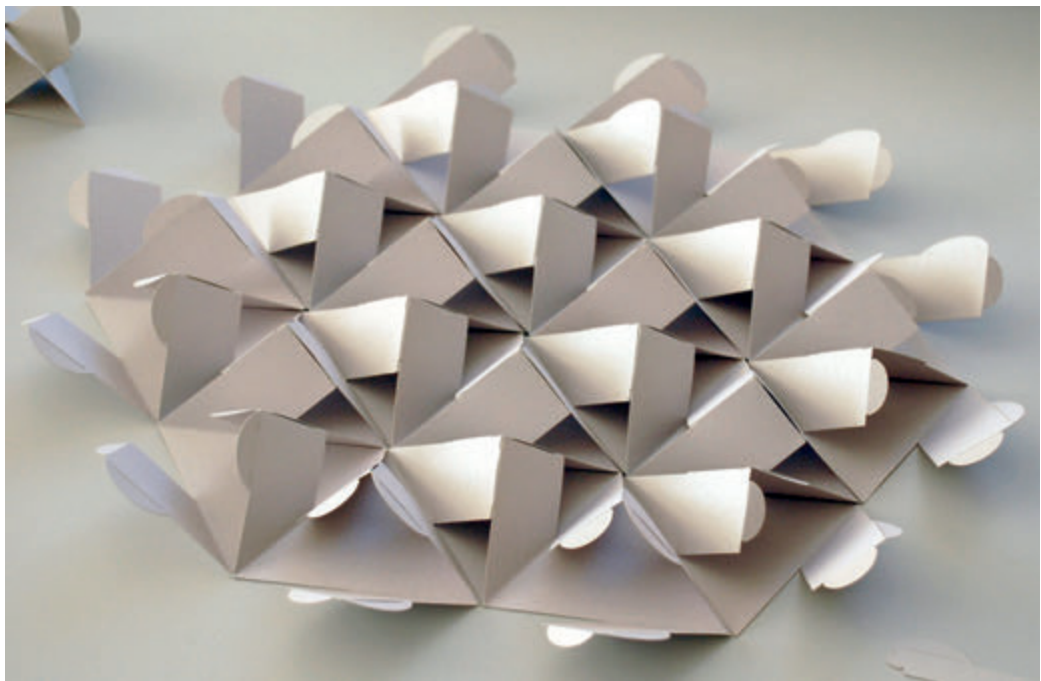
**Figure 24:** *Elevated version of the tetrahelix made with the triangular element.*

The idea of elevation cannot only be applied on polyhedra but also on flat tilings.



**Figure 25:** *Elevation is also possible with plane patterns.*

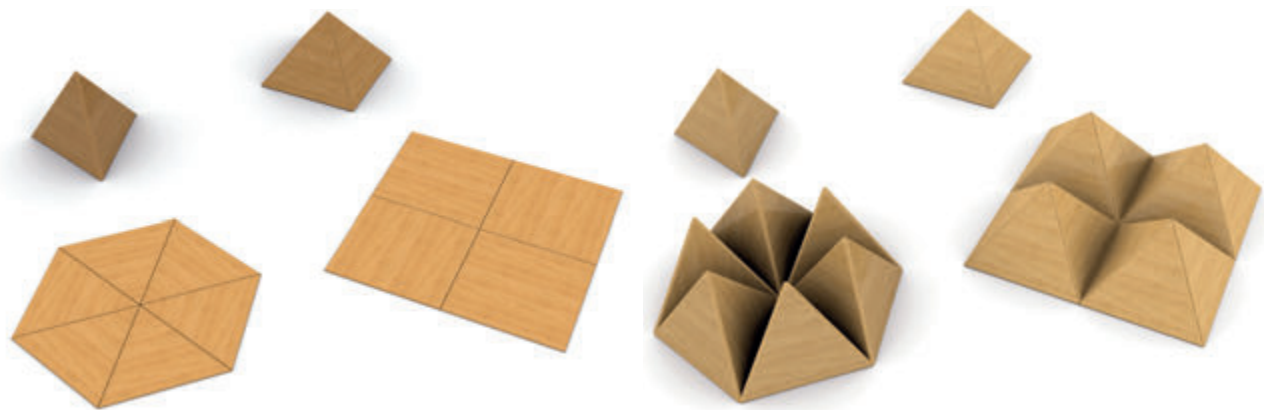
For example, starting with the tiling 3,3,3,3,3,3 in which all the tiles are equilateral triangles, we can put a triangular pyramid on each of the tiles. And when we open up the pyramids like Escher did we can make the model with the triangular elements as is shown in Figure 26.



**Figure 26:** *Model of the elevation of the planar 3,3,3,3,3,3-tiling, built with the triangular element.*

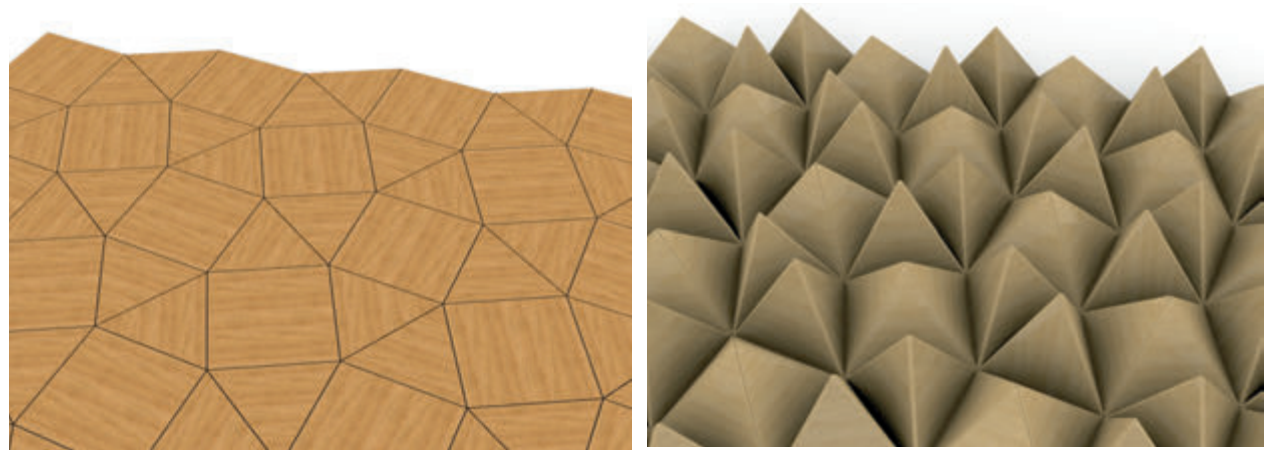


We can apply the same strategy to the tiling with squares. This gives us two regular elevated tilings.



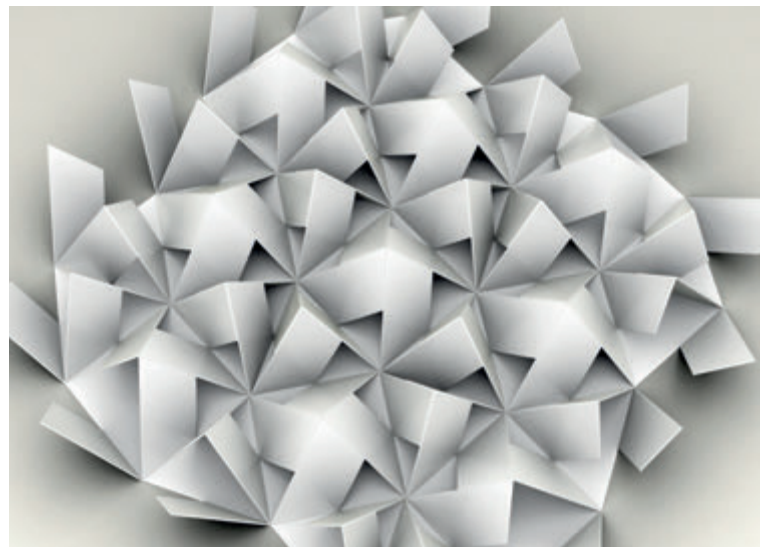
**Figure 27:** *Elevation of triangular and square pattern.*

There is one more Archimedean tiling that we can use in this way: tiling 3,3,4,3,4 shown in Figure 28.



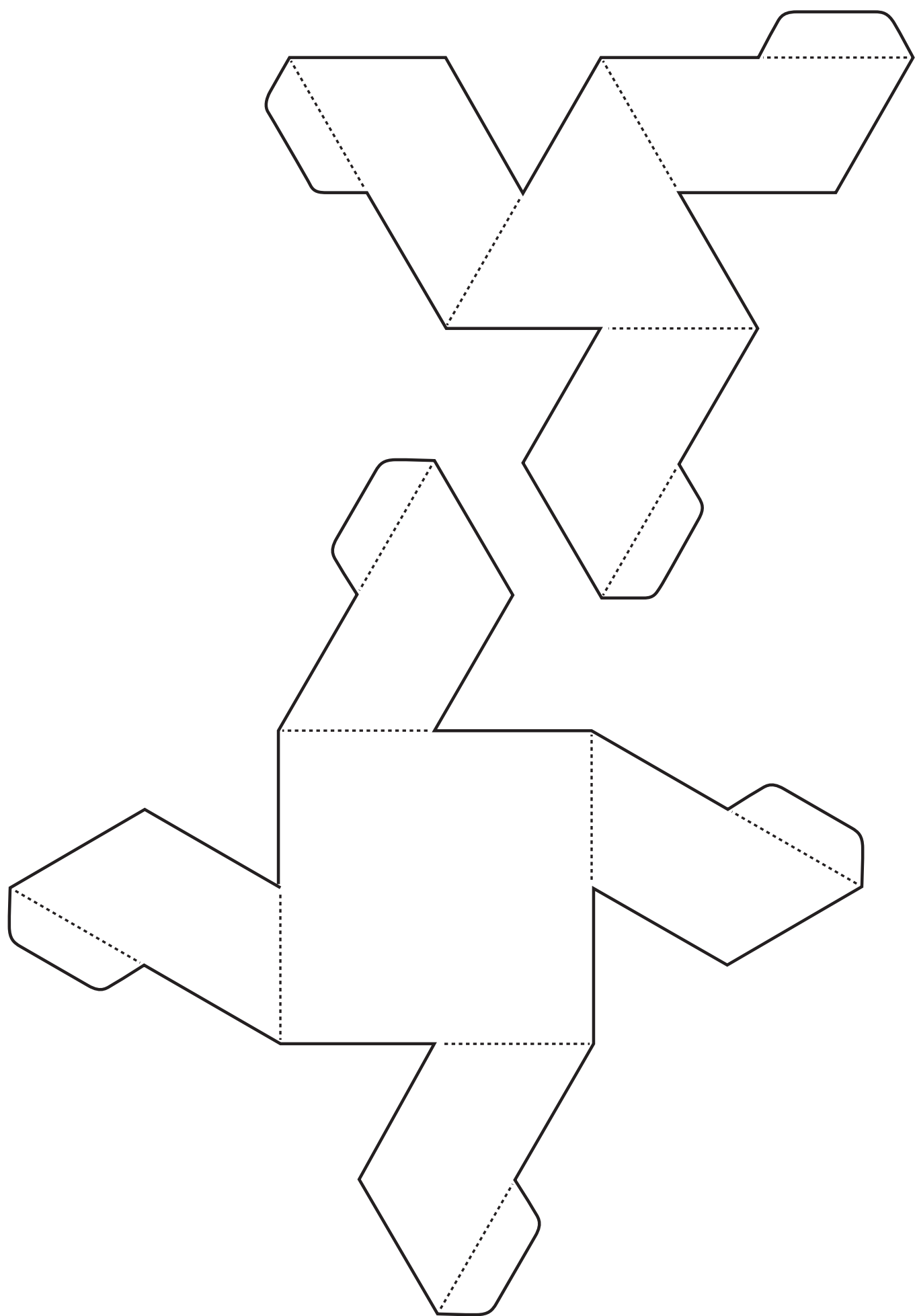
**Figure 28:** *Elevation of the Archimedean pattern 3,3,4,3,4.*

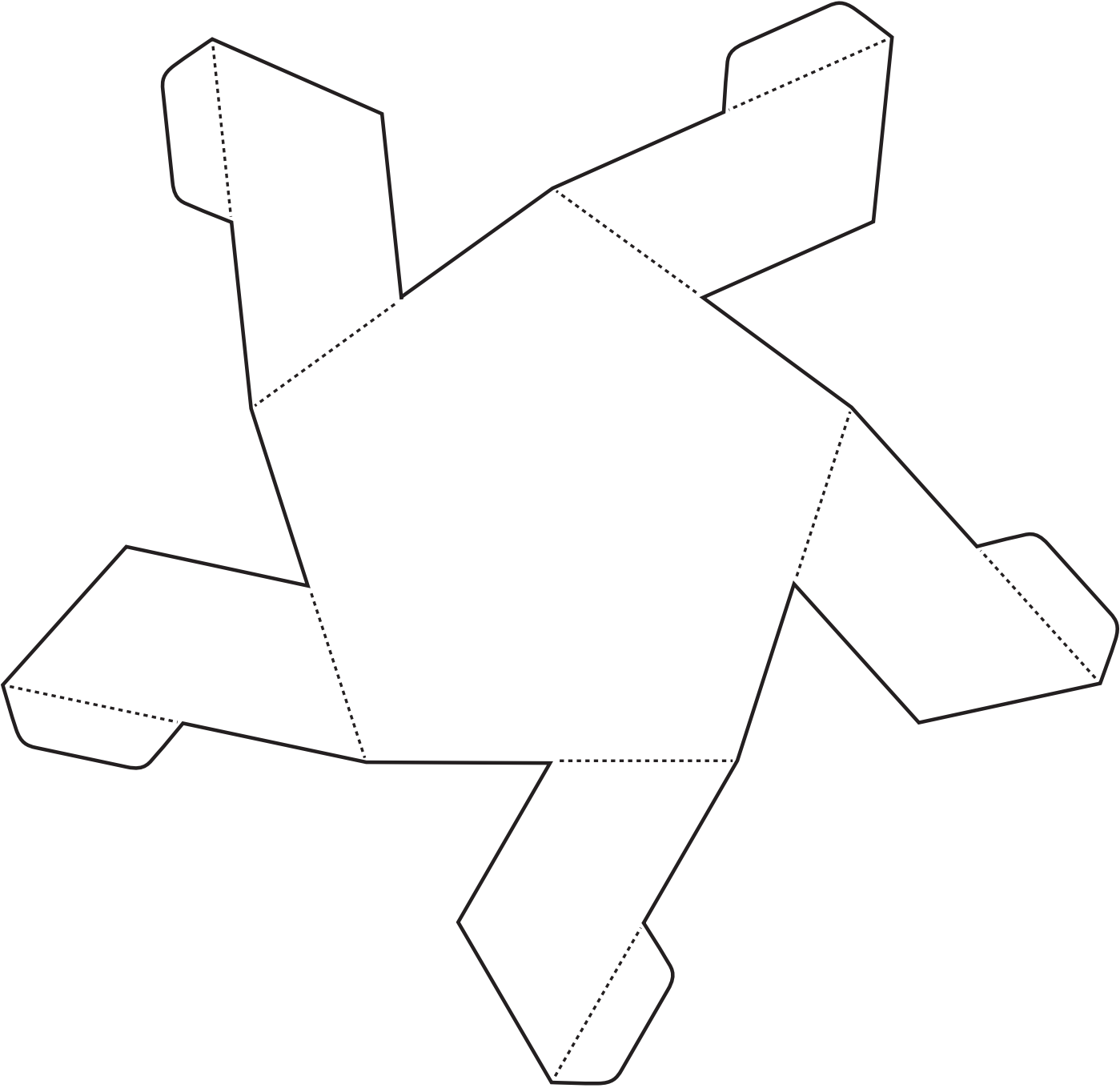
The paper model of the elevation is shown in Figure 29.

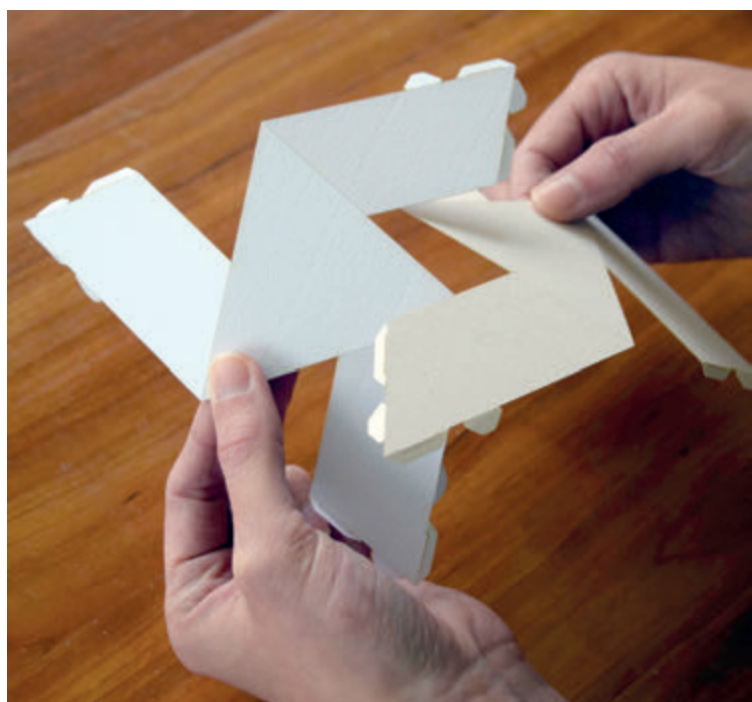


**Figure 29:** *Model built with the triangular and the square elements.*

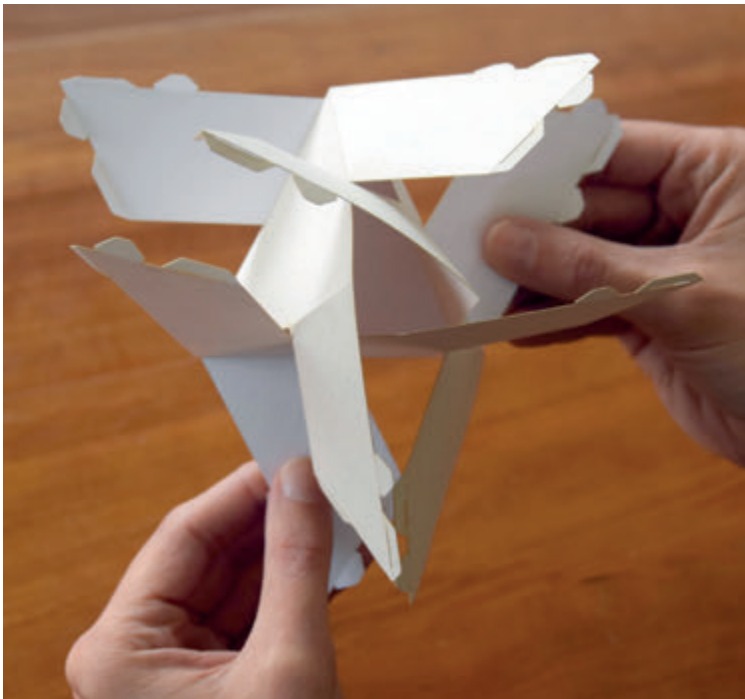
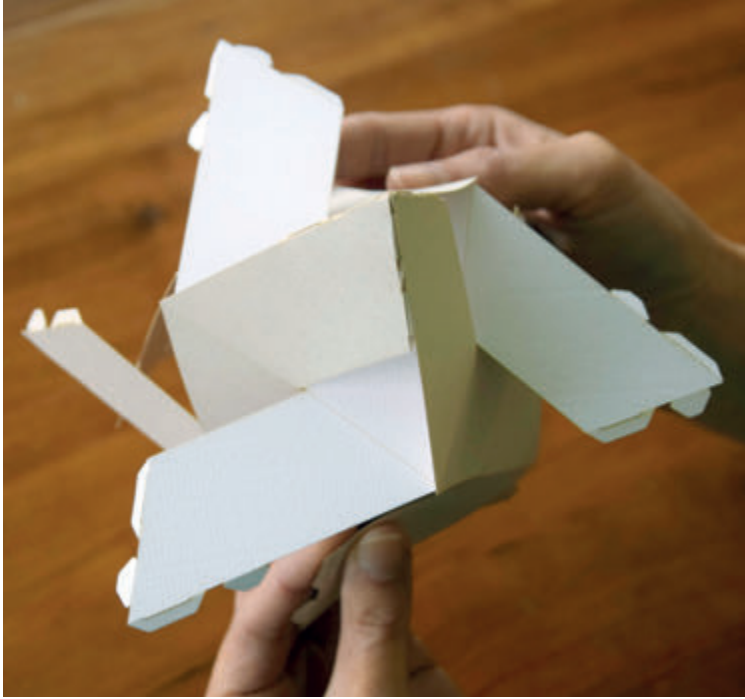
But of course many variations can be made. So with only three different elements we can build a huge collection of interesting mathematical objects and structures.

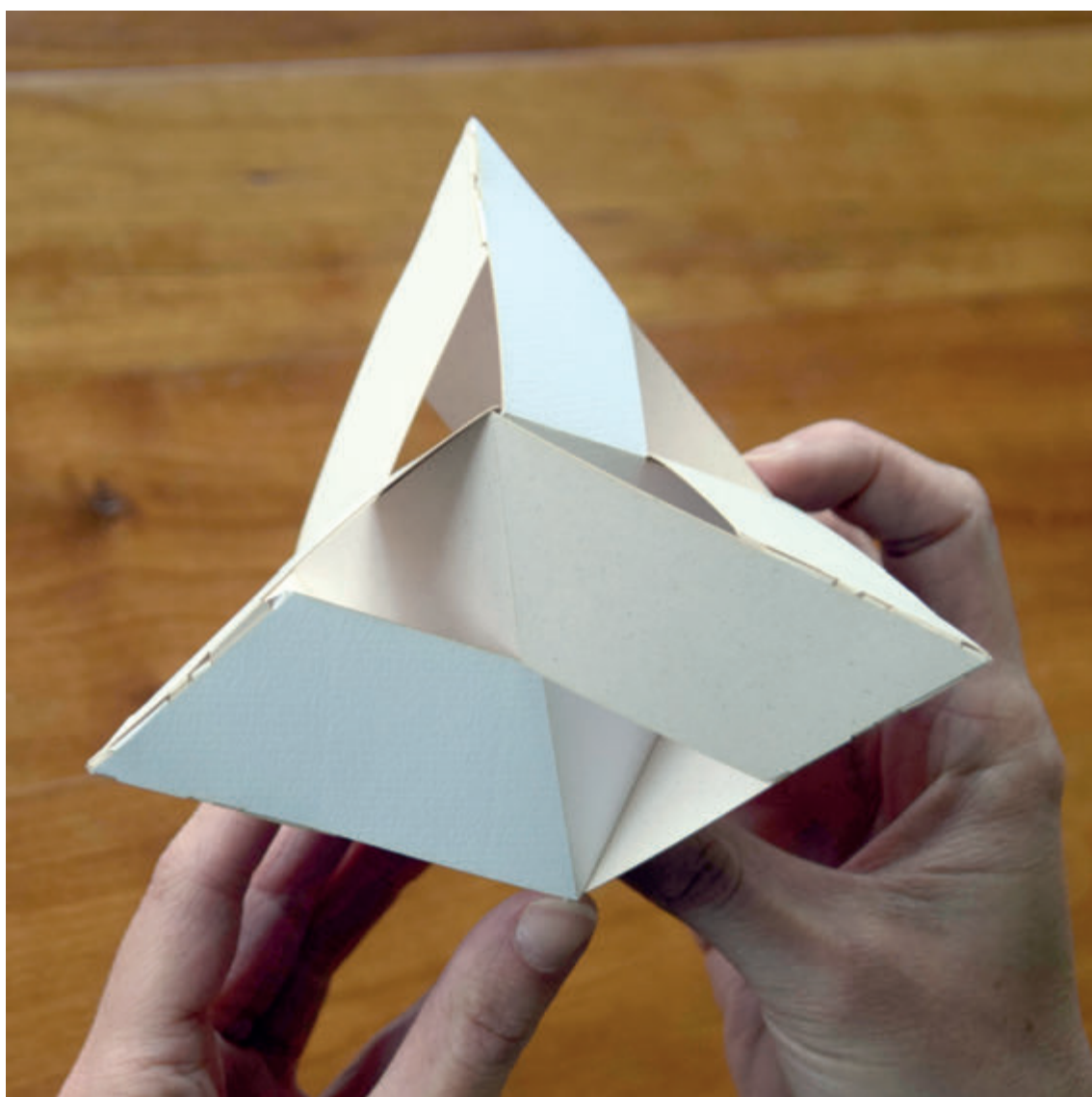














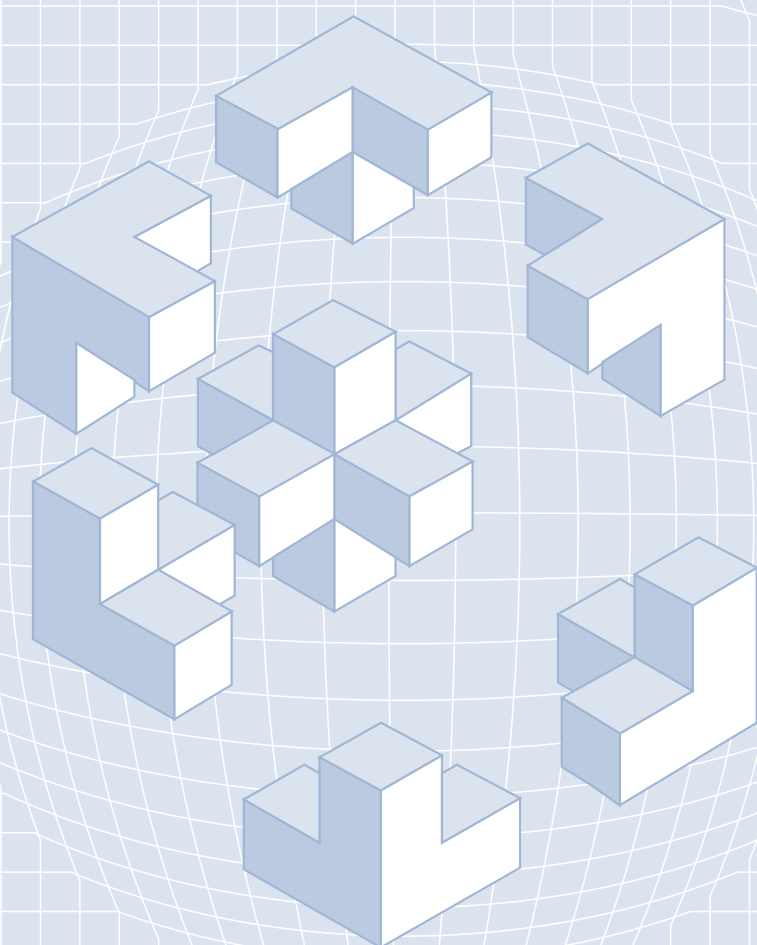
VISUALITY &  
MATHEMATICS

EXPERIENTIAL EDUCATION  
OF MATHEMATICS THROUGH  
VISUAL ARTS, SCIENCES  
AND PLAYFUL ACTIVITIES

FERHAN KIZILTEPE

# TWO EXERCISES ON DIMENSION, SURFACE AND VOLUME IN MATHEMATICS

PAPER SCULPTURES





**FERHAN KIZILTEPE** born, in 1970 in Istanbul. 1989–1993, BA degree in Mathematics from the Faculty of Science, Anadolu University in Eskişehir with a dissertation entitled “Writing of Computer Programme of the Fractal Dimension of Brain Alpha Waves through the Programming Language C”. 1991–1993, associate degree in Pedagogies at Anadolu University. 2008–2011, MA degree in Art and Design Faculty, Yıldız Technical University, Istanbul with a dissertation on “Reflections of the Concept of Mathematical Symmetry as a Method in the Field of Visual and Plastic Arts”. 2010–2011, worked in the Art Faculty at the University Kassel in Germany. Since 1993 she has pursued her postgraduate studies on the relationship between

science and art in various universities including METU, Anadolu University, Hacettepe University, Mimar Sinan University, as well as working in various universities as a visiting professor. She is currently a visiting professor in the Faculty of Architecture and Design at Anadolu University. She was named ‘The most Successful Turkish Woman of the Year 2004’, by the Turkish Women’s League of America in New York, USA; this accolade is just one of the many honor she has received during her career to date.

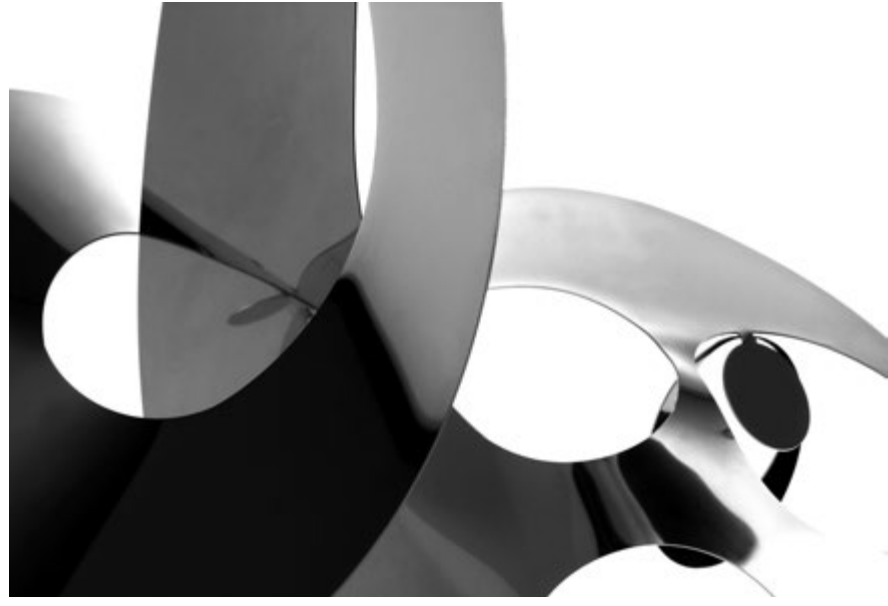
Kiziltepe’s studies are in various fields of art, mostly sculpture and multimedia, along with her theoretical studies on symmetry and related concepts. She has had many national and international exhibitions in which she has showcased her ceramic/metal sculptures and digital prints; presented varied video studies; performed some of her work and created compositions blending all the above. Alongside her creative work, Kiziltepe has organised various workshops introducing basic geometry subjects to children of varying age-groups; seminars and conferences also form part of her educational mission. In 2006, she gave two seminars and two workshops on “Art in Mathematics” for children at the TUBITAK (The Scientific and Technological Research Council of Turkey) festival in Ankara. In 2009, she attended the 11<sup>th</sup> Contemporary Sculptor Association, a group exhibition in Ankara. In 2008, she opened her first solo exhibition entitled “Pre–ColliSculp” at Ars GEometrica Conference and Festival in Hungary. Since 2006, she has also attended many “International Mobil MADI Exhibitions” organised by the Mobil MADI group. See *images below*.

#### Selected publications:

- “A Brief Essay on Jeol- Peter Witkin from Symmetry Perspective”, *15<sup>th</sup> Generative Art Conference Proceeding GA2012*, Italy, 2012. (<http://www.generativeart.com/>)
- “A Brief Review of Frank O. Gehry and the Nationale-Nederlanden Building”, *Bridges Coimbra Conference Proceeding 2011* (ISBN: 97809846064203, ISSN: 1099–6702), USA, 2011, pp. 551–554.
- “Analyzing the Plane Symmetry of Ottoman Tiles as an Architectural Element between the 15th & 16th Century”, *Symmetry: Culture and Science* (ISSN 0865- 4824), vol. 21, No.4, Hungary/ Budapest, 2010, pp. 413–432.
- “Review of a Cinema Film from the Perspective of Symmetry: ‘The Pillow Book’” *Bridges Pécs Conference Proceeding 2010* (ISBN: 97809846064203, ISSN: 1099- 6702), USA, 2010, pp. 439–442.







Figures: Ceramic sculpture (previous page left), paper sculpture (previous page center) and metal sculpture (previous page right), digital prints (top left and right), performance image (bottom left), workshop at the TÜBİTAK (bottom right).

## 1. INTRODUCTION

The two exercises below are designed to provide a period of concrete activity for children at age 11 and beyond. Exercises are arranged so as to engage analytical, creative and practical thinking skills ensuring that the knowledge, comprehension and application stage framework of mathematical education is respected. Also, these exercises will help to connect the basic concepts of mathematics with the plastic and visual arts.

## REFERENCES:

- Bacanlı, H.. *Gelişim ve Öğrenme*. Ankara: Nobel Yayınları. 2002.  
 Saban, A.. *Çoklu Zekâ Teorisi ve Eğitim*. Ankara: Nobel Yayınları. 2002.  
 Alkan, H., M. Altun. *Matematik Öğretimi*. Eskişehir: Anadolu Üniversitesi Yayınları. 1998.

## 2. FROM 2D TO 3D PAPER SCULPTURES

### Exercise I

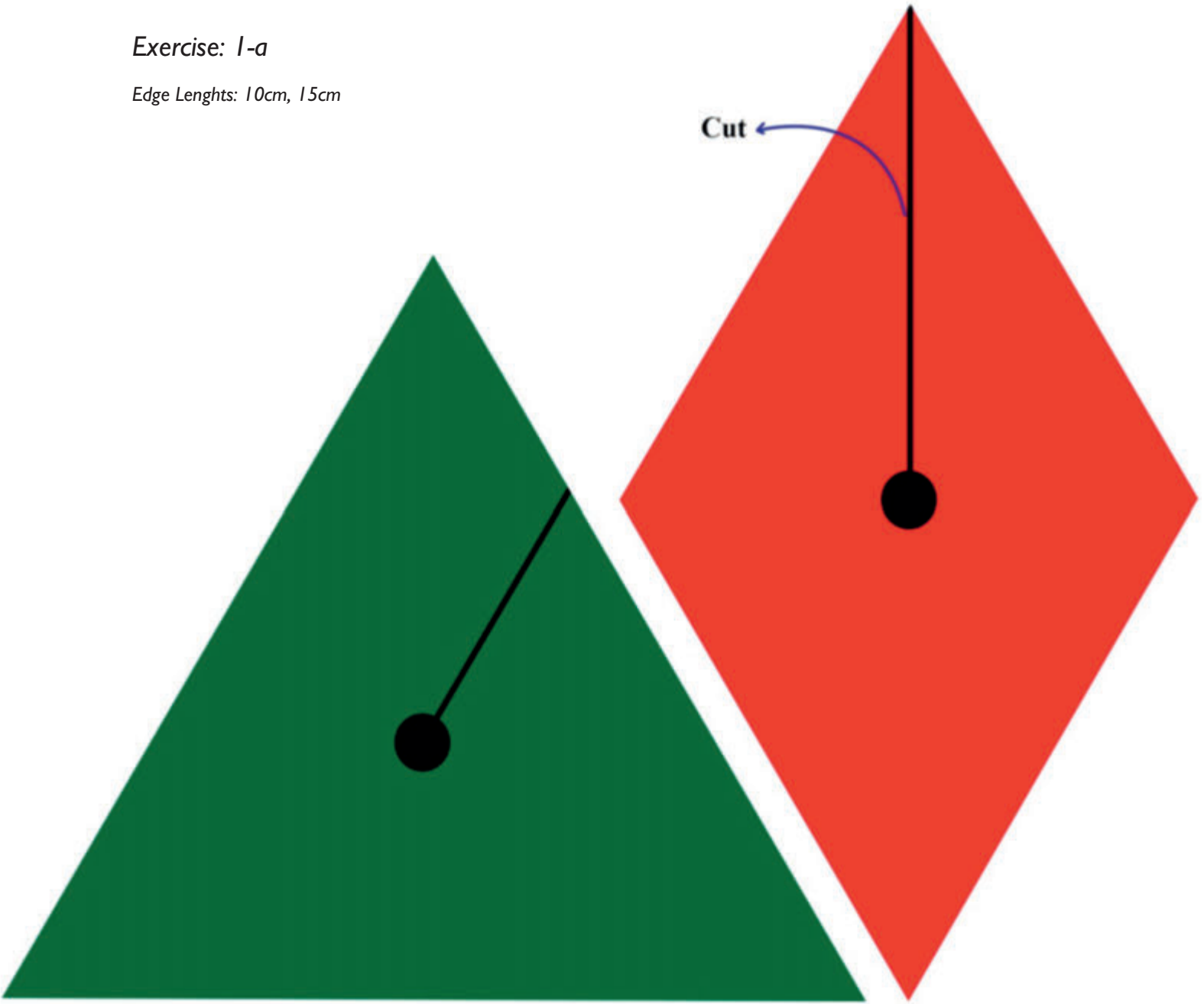
This exercise is designed to assist the students in understanding concepts such as dimension, surface/ plane, volume/ space. Here, by combining surface (2 dimensions, 2D) pieces (as outlined below) together participants will create a self-standing object that is a 3 dimensional (3D) object. In other words, while going from 2D to 3D there will be an object which has volume created from surfaces. Thus the target group will observe that 2D's geometry can only be seen, whereas 3D's geometry can be both seen and touched. Also the plane of 2D and volume of 3D can be equated. It is important to remember that the dimensions of the paper sheet being used to construct the structure are unimportant, participants need to focus on its surface, nothing else.

The material used in the exercise is paper. It should be at least the weight of copy-paper. If the paper's weight reaches 200 gr/m<sup>2</sup> (or more), the finished object will be more durable. It is thus advised to undertake trial runs with lower gram papers. It is also advised to use coloured paper and the dimensions of paper used in the exercise should be a min. 5x5 cm – max. 20x20 cm (for over 200 gr papers, 30x30 cm is optimal). For younger children, paper should be pre-cut; older children should be supervised using scissors so as to avoid accidents. In each exercise the two pieces of paper should be joined at the incisions and glued together, preferably with paper adhesive or another method of the children's choosing; at this stage of development it is very important to allow children to use a different method of folding or connecting. Other pieces of paper can also be added to the mix and if they are, the design will become increasingly complex. Paper shapes can consist of circles, ellipses, rectangles or polygons; children can use the shape they want to. If necessary, at the end of the study, the finished sculptures can be tidied up with scissors...

The following images consist of 1) Samples of paper forms 2) Ready-to-use paper-cuts 3) Photographs of completed paper sculptures.

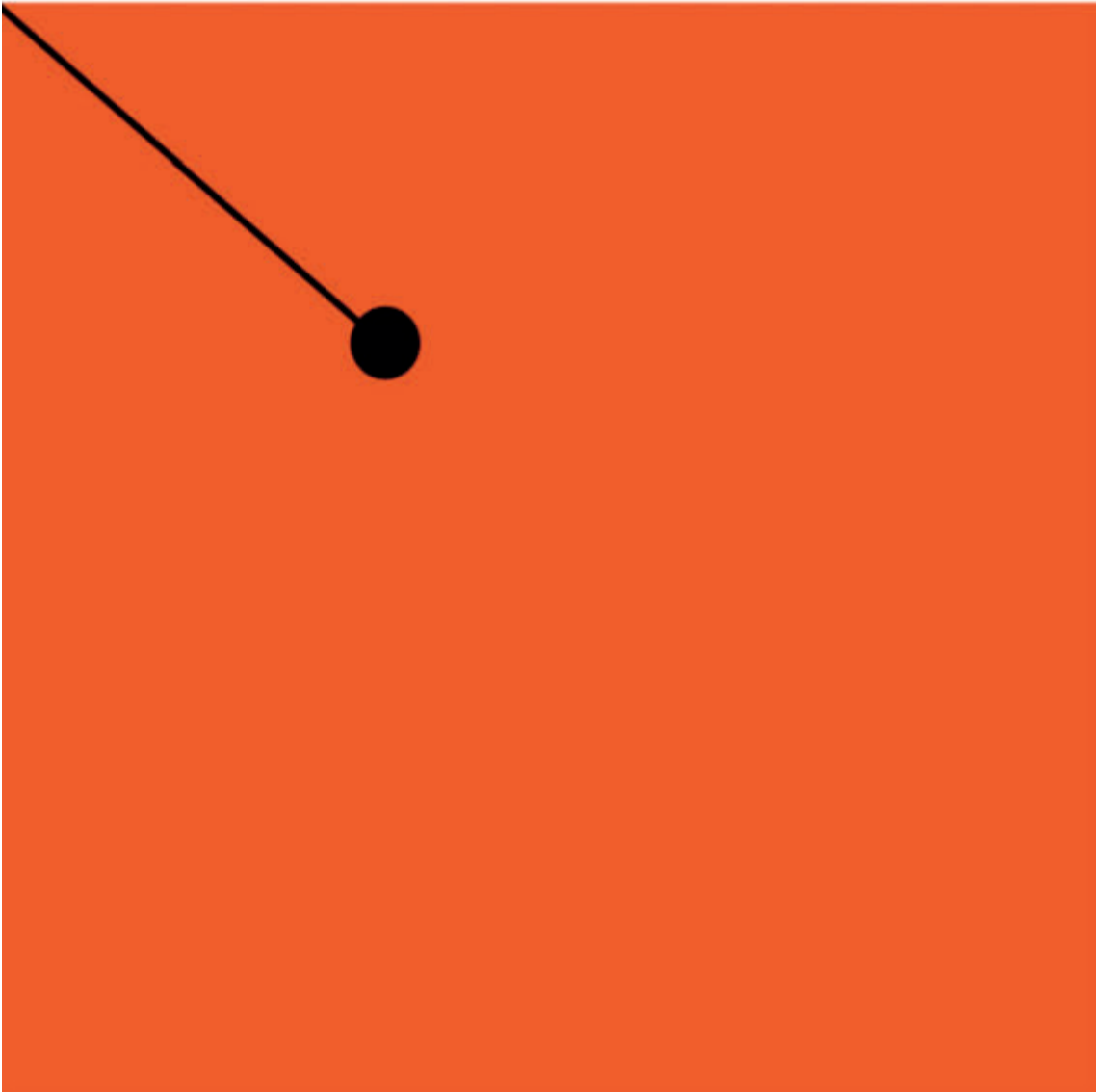
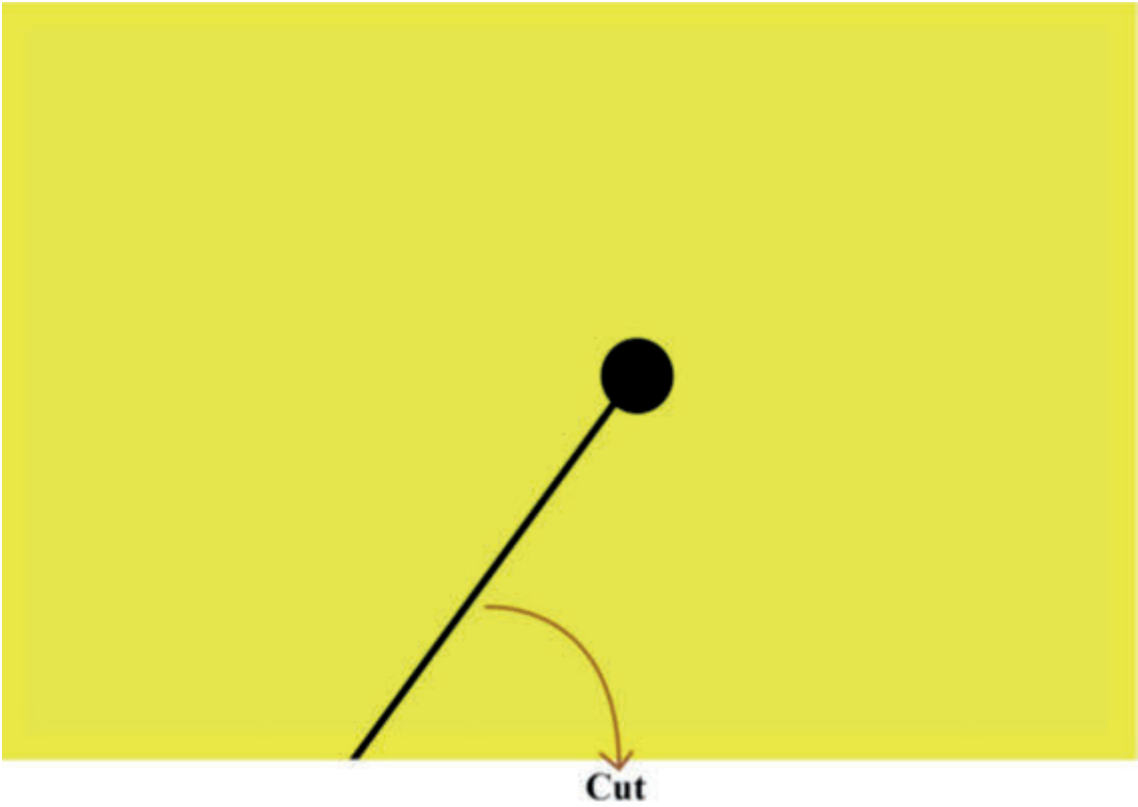
*Exercise: I-a*

*Edge Lengths: 10cm, 15cm*

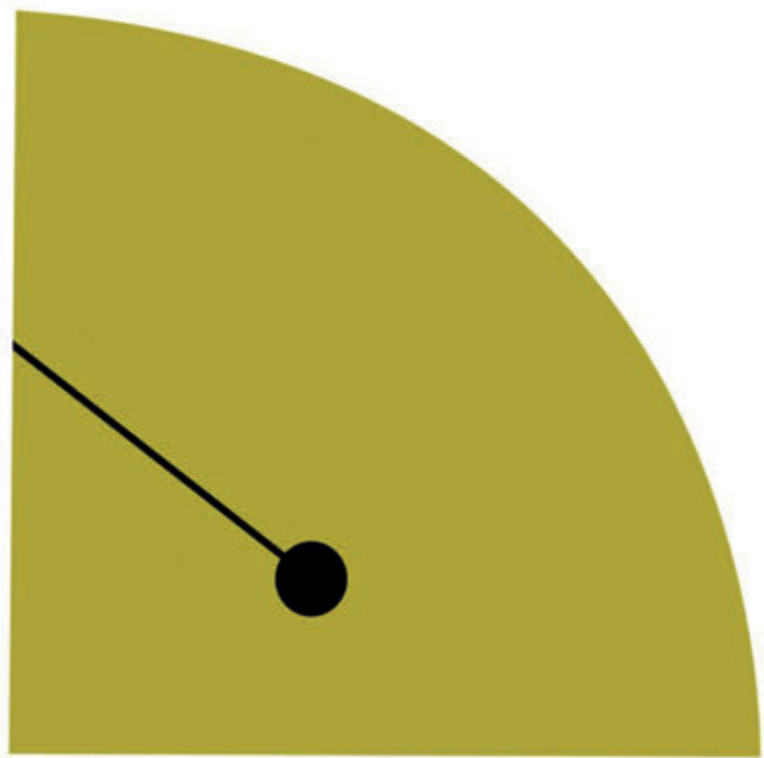
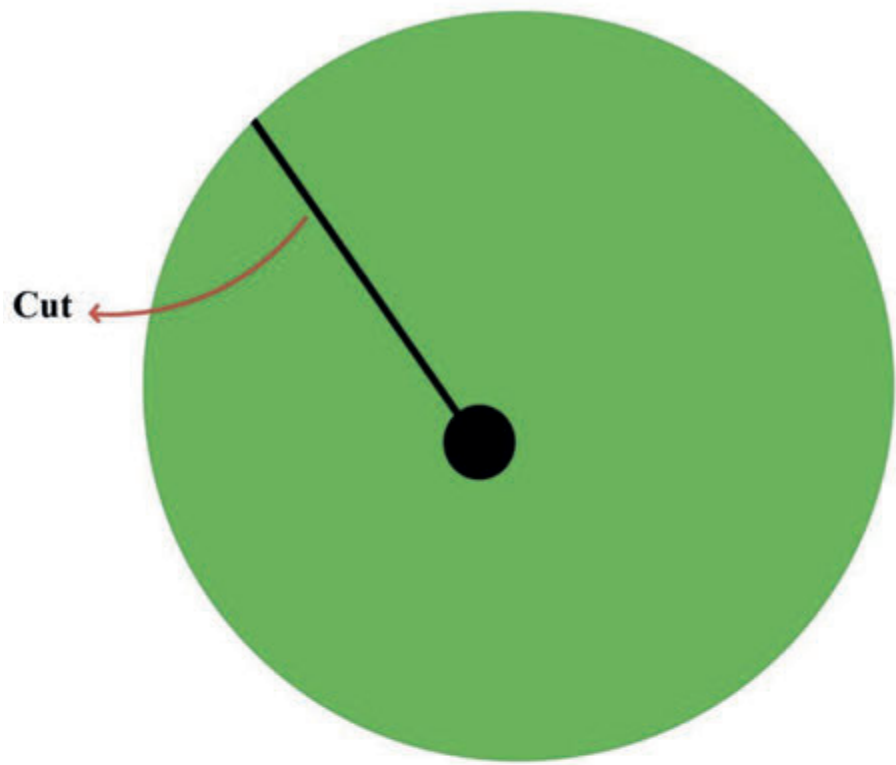


*Scale: 1/1*

Exercise:  
I-b



Exercise: I-c



Diameters: 10cm, 20cm

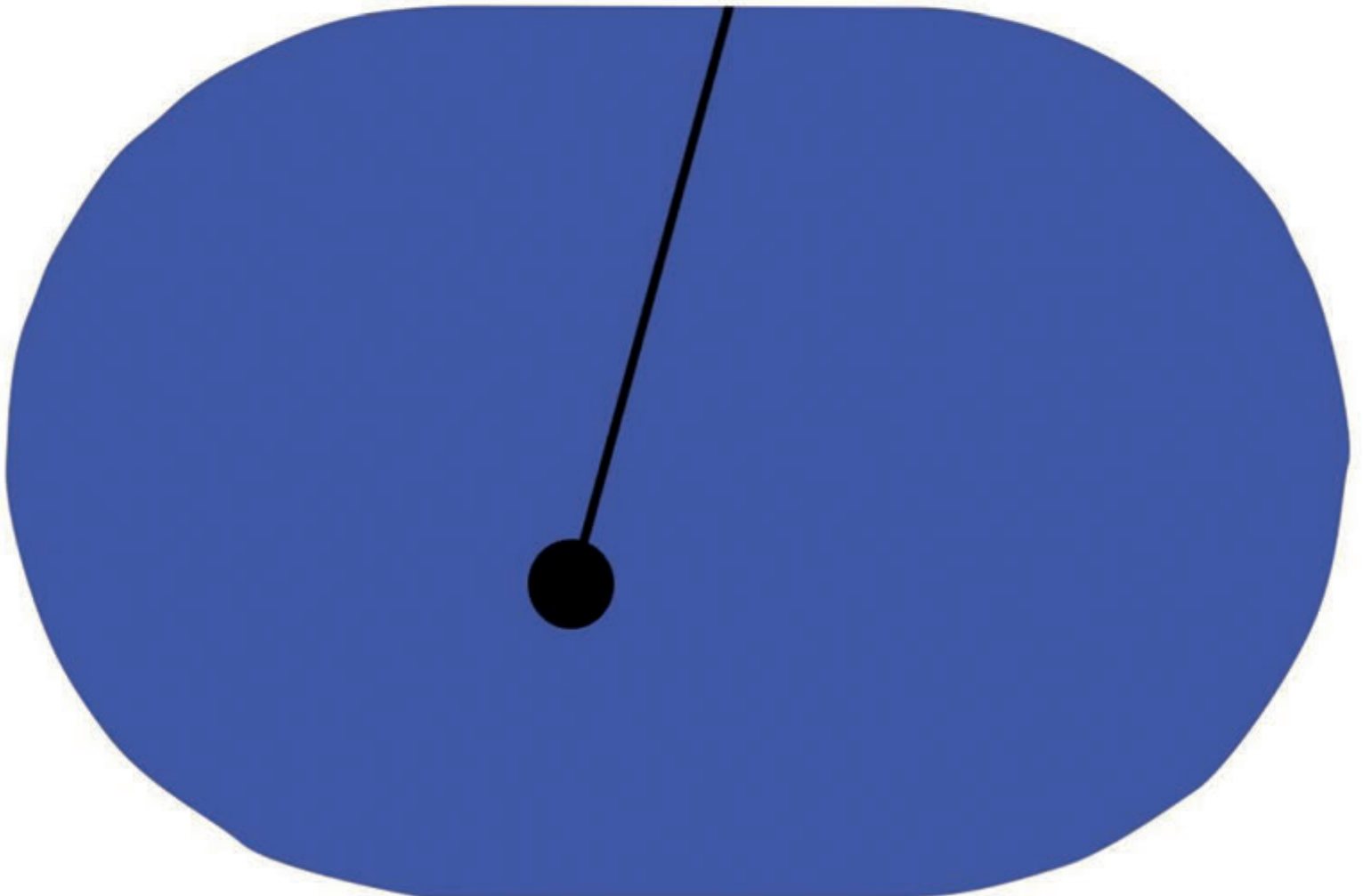
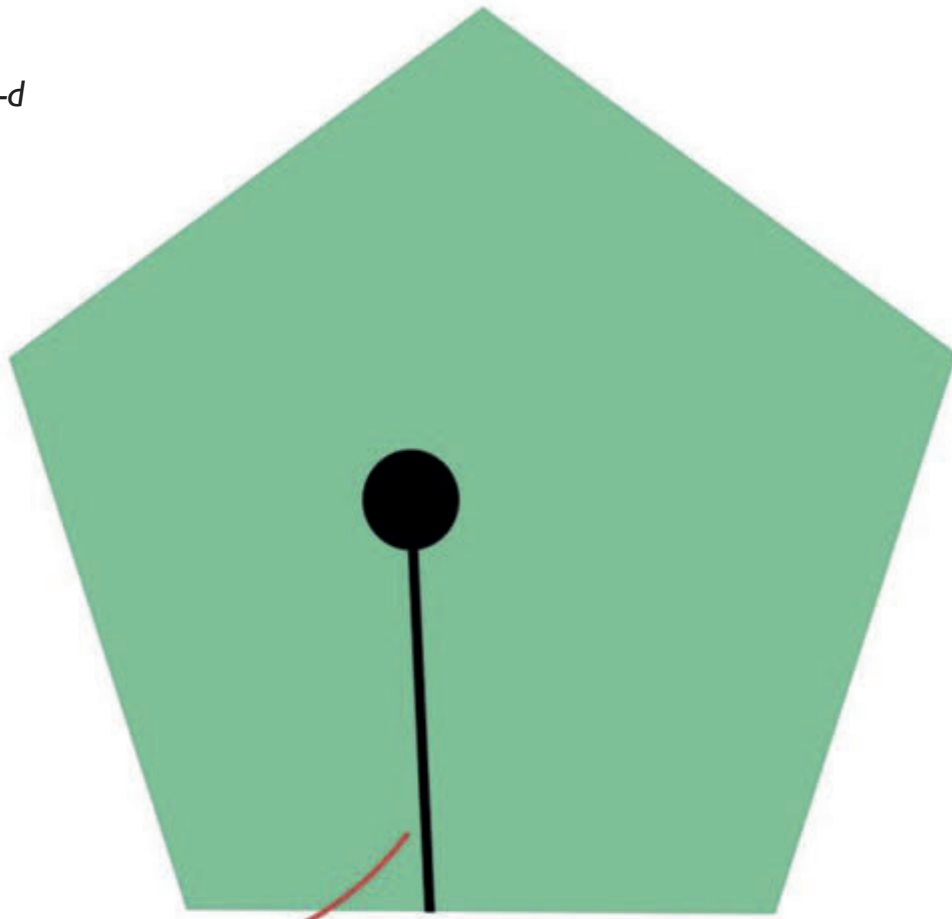
Scale: 1/1



Exercise: I-d

Diameters: 10cm, 20cm

Scale: 1/1





Example 1.1



Example 1.2



Example 1.3



Example 1.4

The paper sculptures in the examples, were made with Canson 200gr colorful tracing paper.

### 3. FROM NEW SURFACES TO NEW VOLUMES

#### Exercise 2

This exercise which has the same educational goals as the previous one, aims to create new surfaces and new volumes from basic regular 3-D solids; generating new surfaces from regular dimensions and new volumes from new surfaces will be explored. This will aid the development of discrete thinking skills and creativity, as the children create new surfaces and volumes their appreciation of discrete sculpture will increase. Thus, as a result of the exercise, the participant's perception of coloured volumes' paper sculptures will grow.

The main material used in this exercise is paper. In the first part of the task white paper will be used, in the second, coloured. The paper should be, at the very least, the diameter of copy paper. If the paper's weight is 200 gr/m<sup>2</sup> (or more), the finished sculpture will be more durable. It is thus advised to undertake trial runs with lower gram papers. It is also advised to use min. 8x8cm – max. 20x20 cm (for over 200 gr papers, 30x30 cm is ideal). The coloured paper will need to cover the new surface of the white paper volumes, thus, it is recommended to use at least 60 – 110 gr coloured paper. For younger children, paper should be pre-cut; older children should be supervised using scissors so as to avoid accidents.

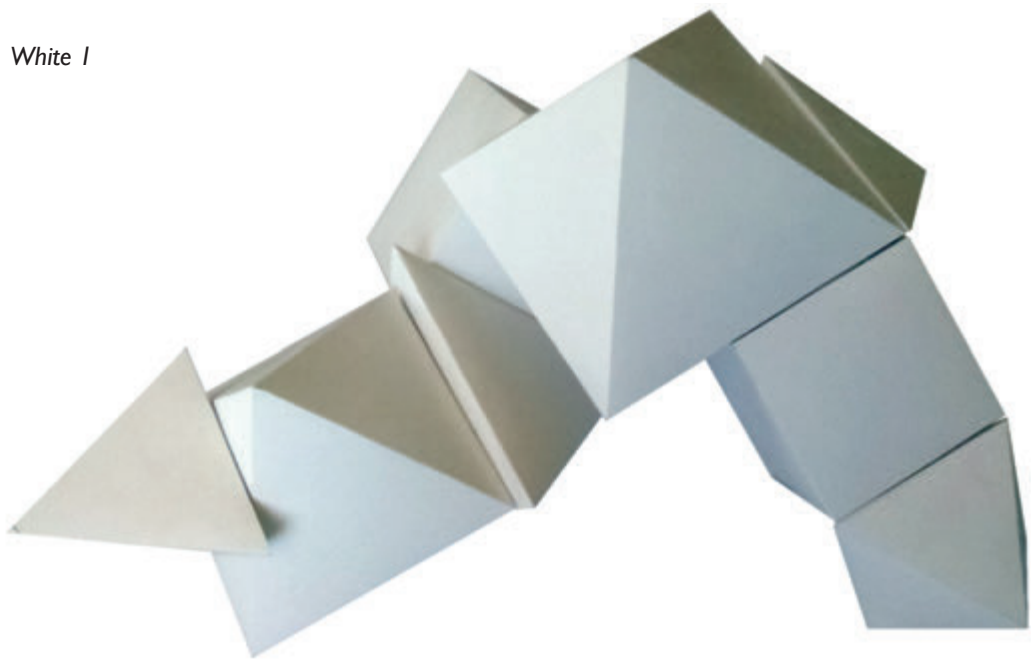
In the first part of the task polyhedra are formed from white paper; among these are various base, straight or oblique prisms, pyramids or octahedron (eight faces), dodecahedron (twelve faces), icosahedron (twenty faces), etc. At least two polyhedra should be used in the exercise. White paper should be cut to the chosen polyhedron and then

the edges folded and pasted to each other. Prepared polyhedra should be affixed at their chosen faces or, to create the impression of one passing through the other, they should be connected at various angles. In this way a new polyhedron will be formed by combining the existing polyhedra. The task will become more difficult if attempted with many-faced polyhedra, and this will expose participants to more complex problems. The basic regular polyhedra below are drawn in their open versions.

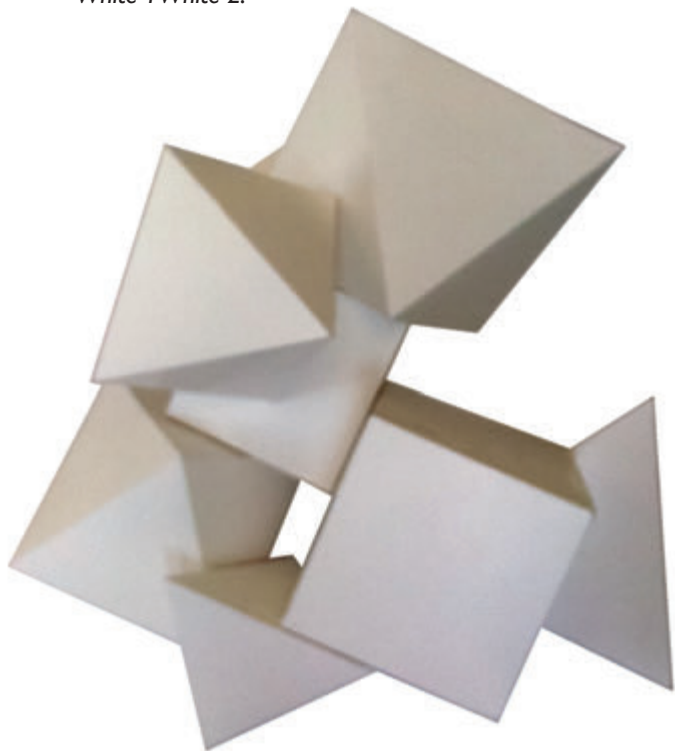
The faces of the white polyhedron, once covered with coloured paper will now become a new volume; it is now a coloured paper sculpture.

The following images are of 1) New polyhedra made with white paper 2) Samples of paper sculptures created by covering with coloured paper 3) Open versions of basic regular polyhedra .

*White 1*

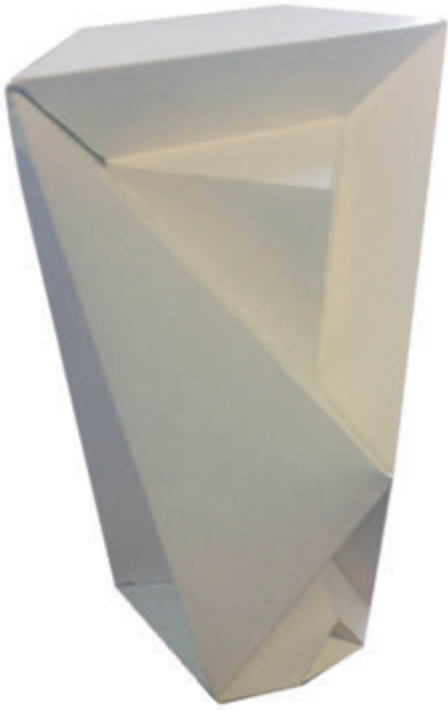


*White 1 White 2.*



*White 1 White 3.*

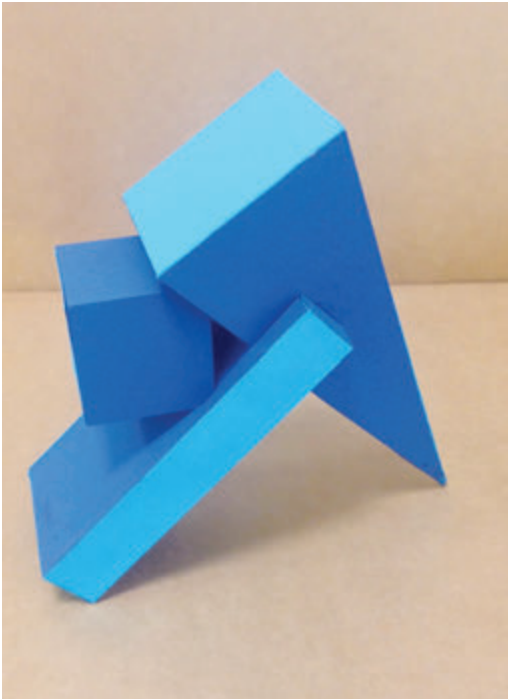




White 4  
White 5  
White 6

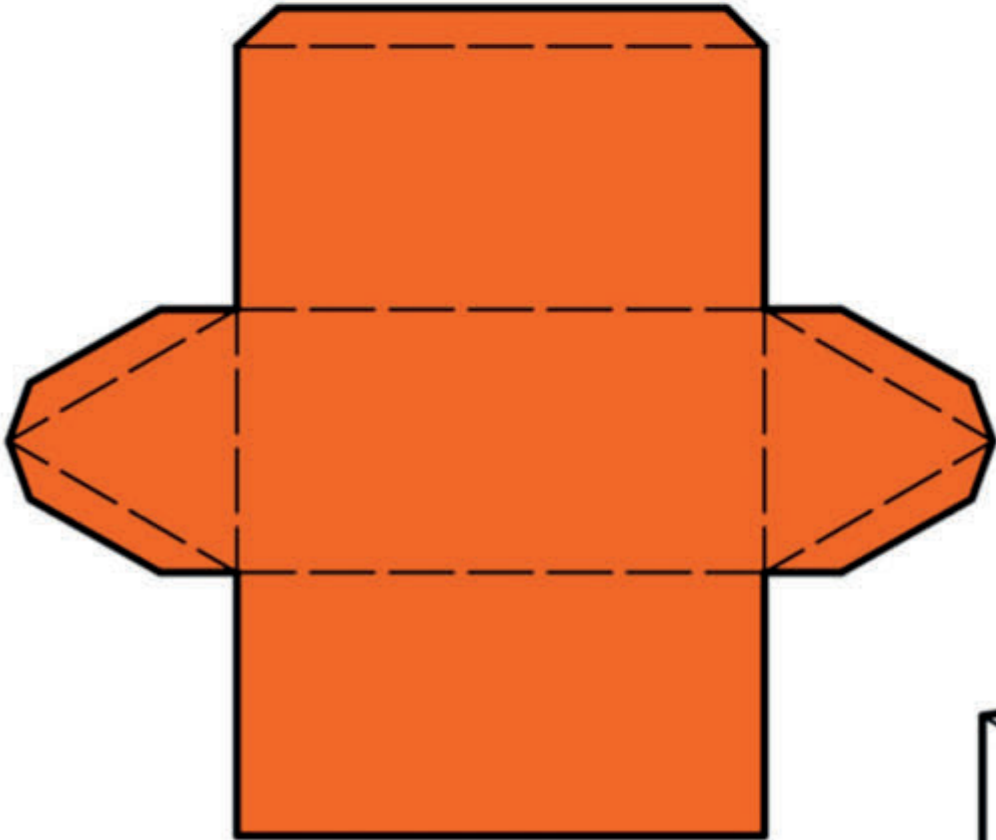
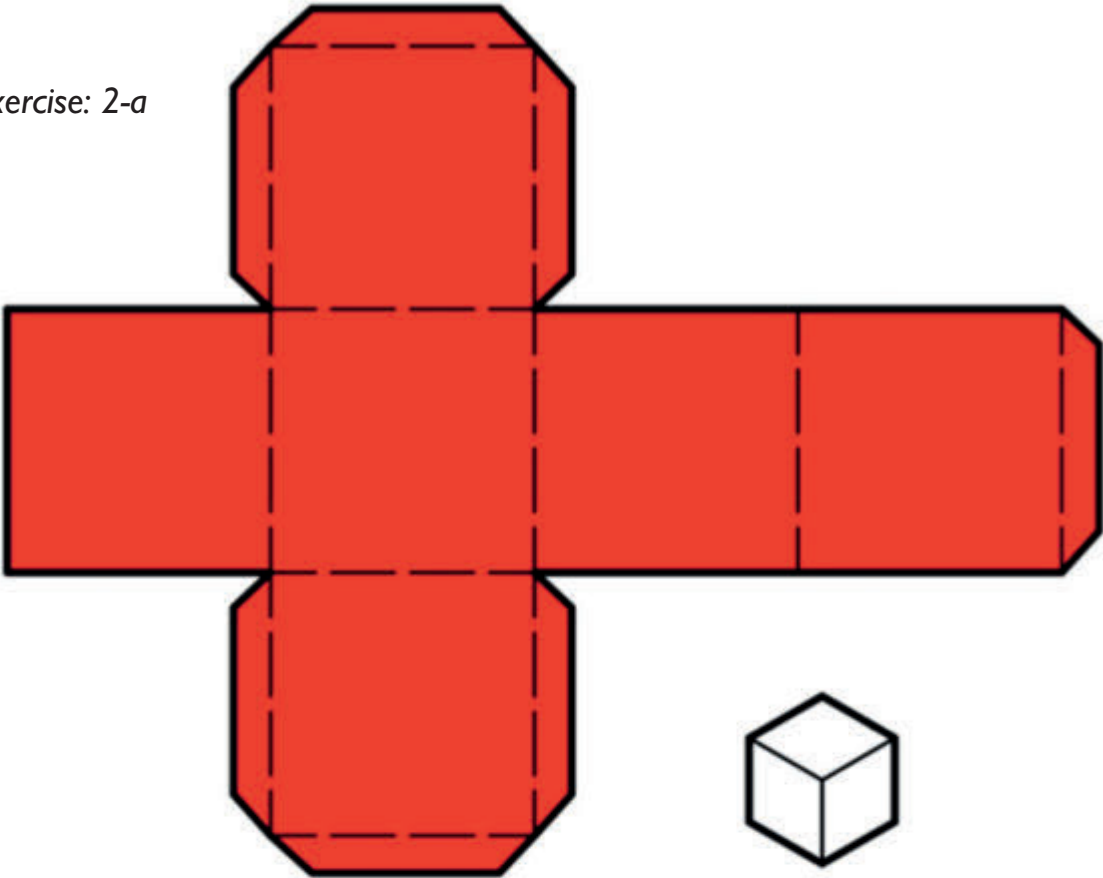


Coloured 1 Coloured 2, Coloured 3





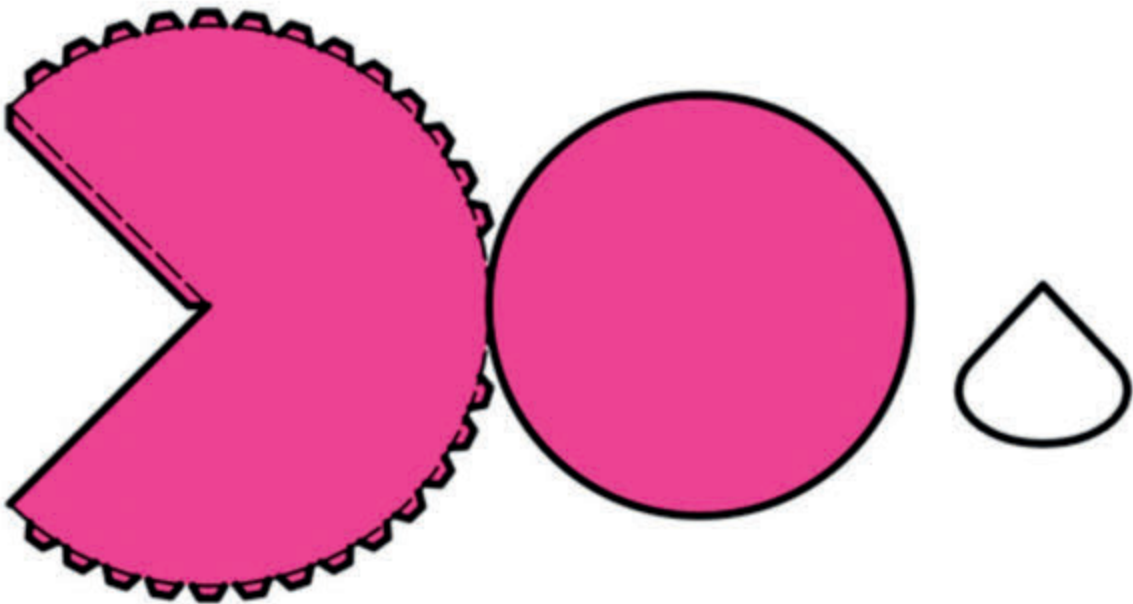
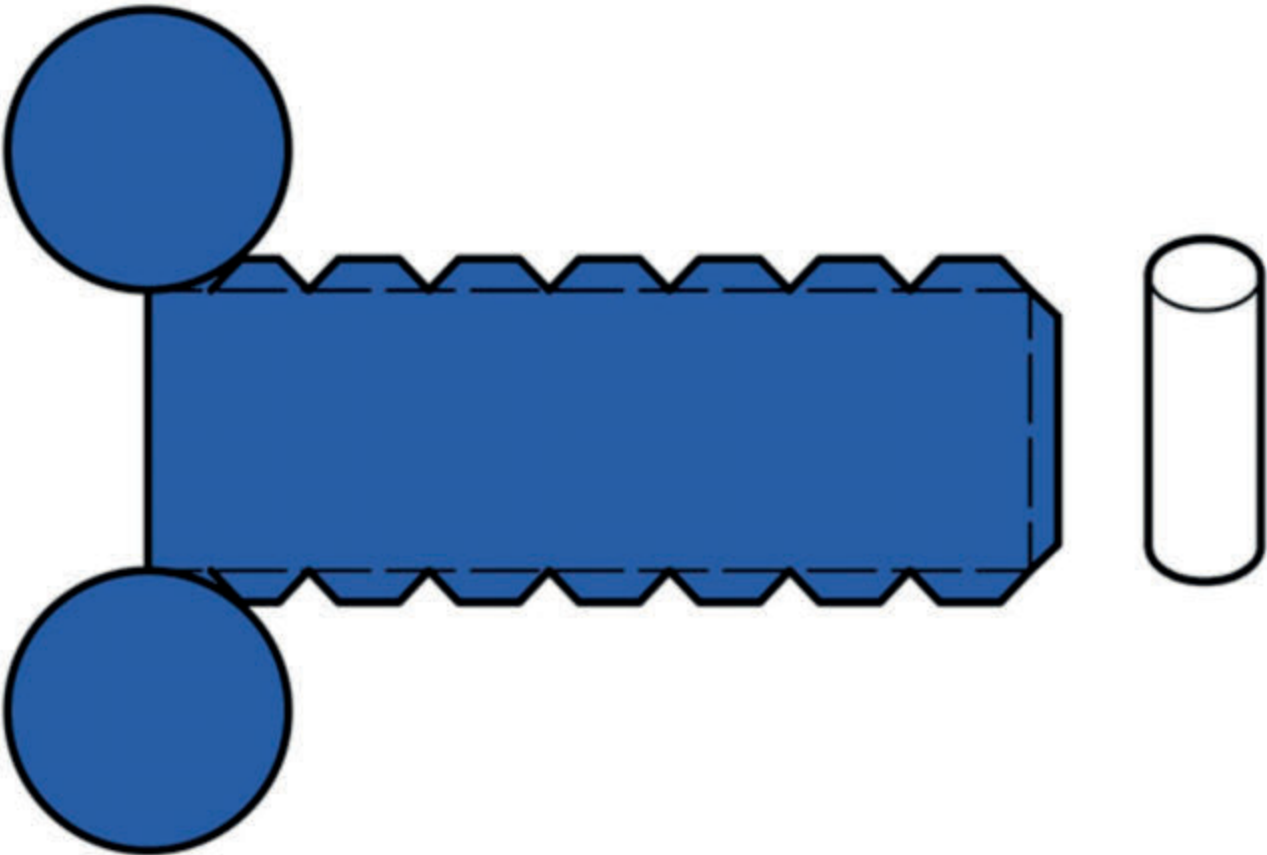
Exercise: 2-a



Edge Lengths: 10cm, 20cm

Scale: 1/3

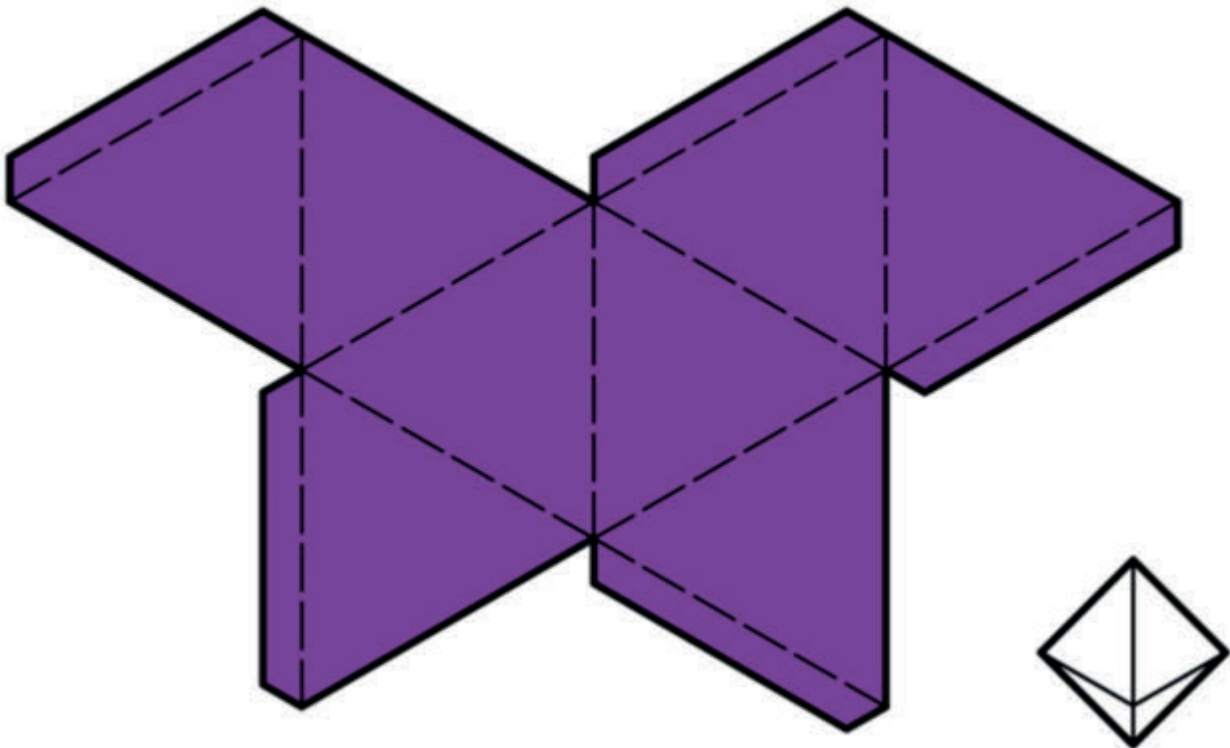
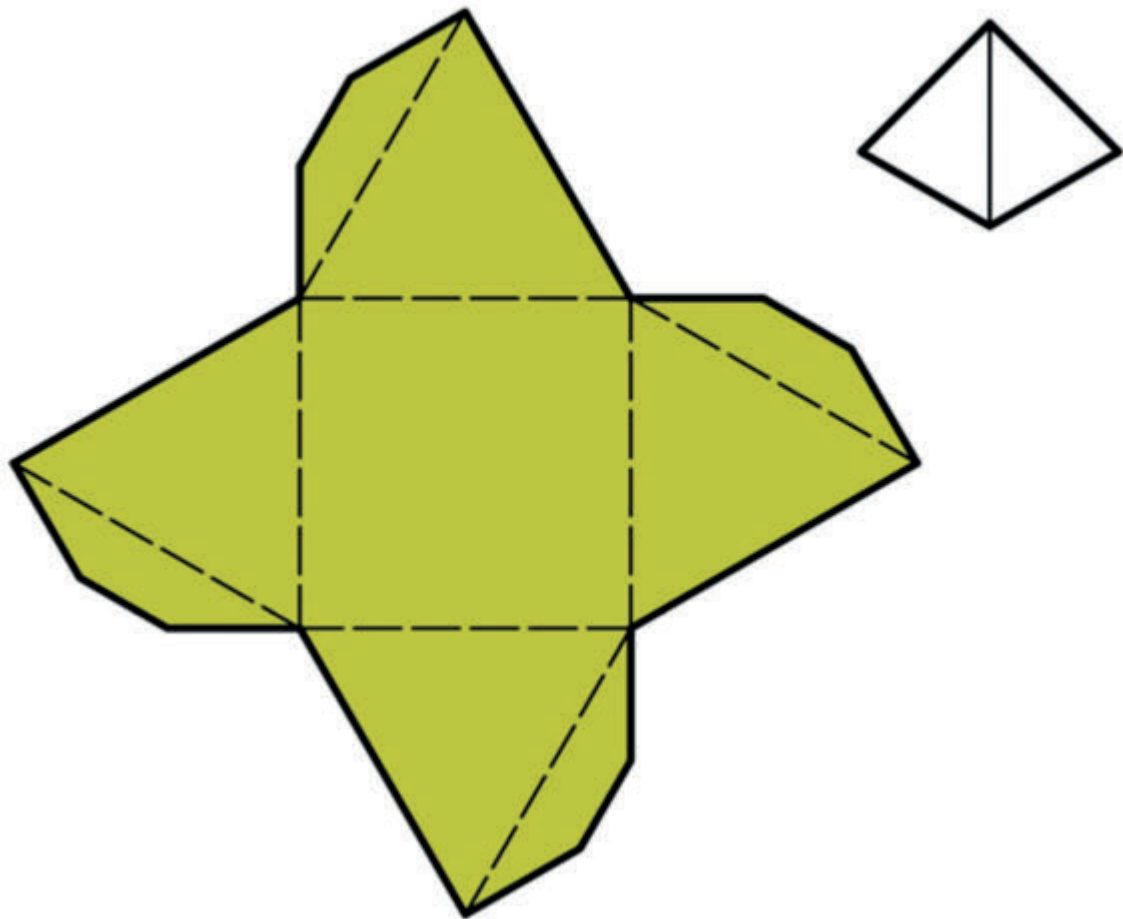
Exercise: 2-b



Diameters: 20cm, 15cm, 10cm

Scale: 1/3

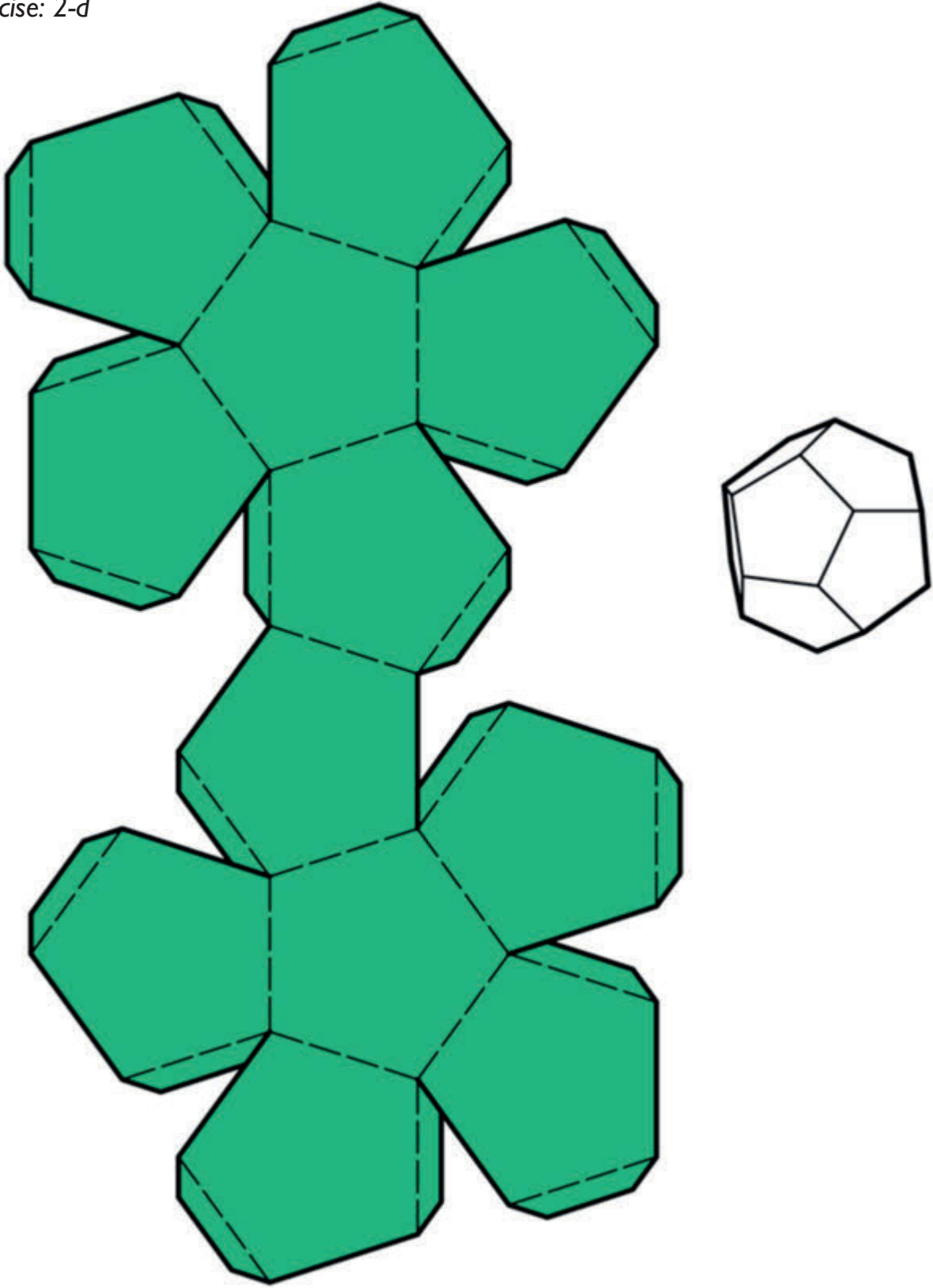
Exercise: 2-c



Edge Lengths: 17cm

Scale: 1/4

*Exercise: 2-d*



*Edge Lengths: 12cm  
Diameters: 10cm*

*Scale: 1/4*

Thanks for the contributions of philologist Melike Çakan and industrial designer Ramazan Seyhan.





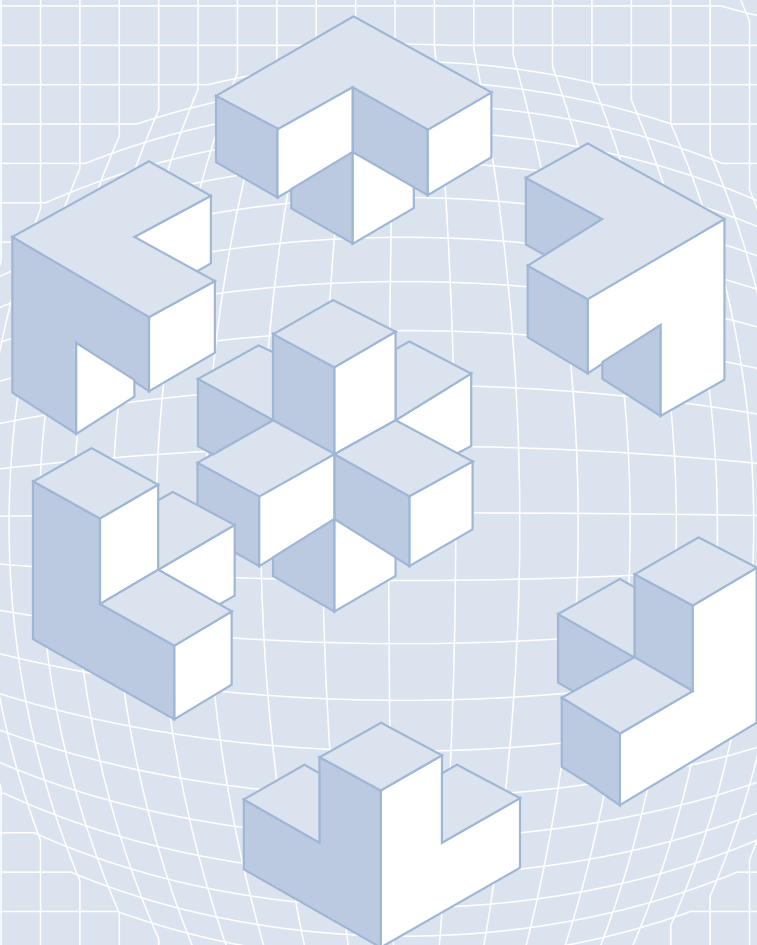
**VISUALITY &  
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KRYSZYNA BURCZYK - WOJTEK BURCZYK

# MATHEMATICAL ADVENTURES IN ORIGAMI

PAPER SCULPTURES



**KRYSZYNA BURCZYK** and **WOJTEK BURCZYK** graduated in pure mathematics from Jagiellonian University in Krakow, Poland in 1983. Wojtek also holds a Ph.D. from AGH University of Science and Technology in Krakow. Krystyna is both a math teacher with more than 20 years' experience and an origami artist. Her mathematical background has led her to develop an interest in geometric models. The mathematical structure of origami models and the folding process, as well as the relationship between origami and mathematics has held a particular fascination for her since university. She is also interested in the educational application of origami, especially teaching mathematics through origami. Wojtek was a chief financial officer for a large multinational manufacturing company for more than 20 years.



**Contact details:**

Krystyna Burczyk: burczyk@mail.zetosa.com.pl  
 Wojtek Burczyk: woj.burczyk@gmail.com  
 Website: www.origami.edu.pl

## OUR INTEREST IN ORIGAMI

Krystyna and Wojtek's interest in origami began in 1995 and has now become a large part of their lives. They have exhibited their work at several conventions and exhibitions in Poland and abroad (Hungary, Italy, UK, USA, Spain, Sweden, Germany, Romania, Switzerland, France, Holland and Singapore). They were invited to participate in the large international origami exhibition *Masters of Origami* in Salzburg (2005) and Hamburg (2007), and *Folded Paper: Infinite Possibilities*, an exhibition which opened in Los Angeles in 2012 and will travel around the U.S. until 2016. Krystyna's works were also exhibited at the Art Exhibition of Bridges Mathematics, Music, Art, Architecture, Culture conference in Pécs, Hungary (2010), Coimbra, Portugal (2011) and Enschede, The Netherlands (2013) and Seoul, Korea (2014). Her work can also be found in Nick Robinson's *The Encyclopedia of Origami*.

Krystyna and Wojtek have presented on the theoretical and educational aspects of origami at the Origami Science Math and Education (OSME) conferences in 2001, 2006, 2011, and 2014, the *Didaktik des Papierfaltens* conferences from 2006 to 2013 and the Bridges conference in 2011.

## 1. ORIGAMI AND MATHEMATICS

Origami is about folding along straight lines. Such lines lead us directly into the world of geometry. A simple sheet of paper can become a powerful math tool and, even better, the math problems generated by origami are real-life problems, as opposed to most of the problems students face in the conventional classroom.

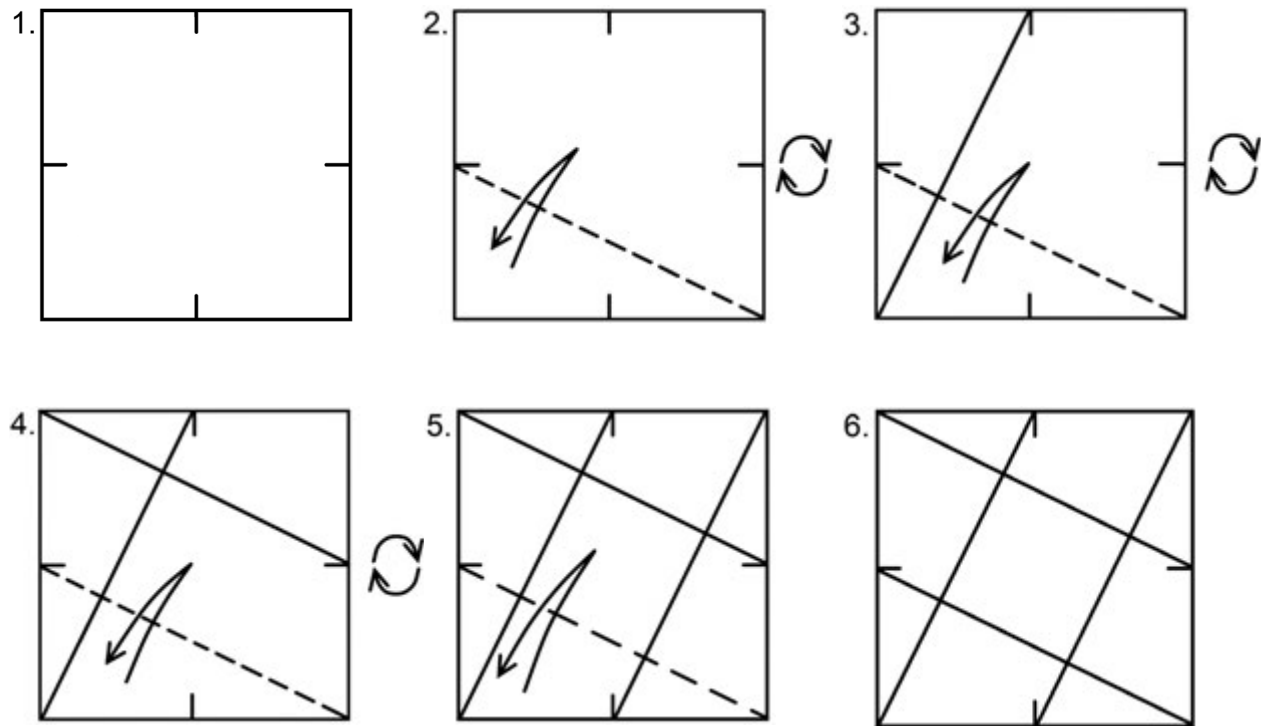
The following activities are based on simple folding and do not require any previous experience in origami or special skills. Diagrams are based on the widely used notation, should any problems arise, consult the short explanation of the symbols at the end of the first activity.

Both activities are original ideas developed by Krystyna, each starts with a detailed description of the folding process leading to a model. This is followed by a series of questions exploring mathematical problems that naturally emerge from the folding process; the solutions open up new avenues for exploring different variations and, ultimately, lead towards art.

## 2 DIVIDE A SQUARE INTO 9 PIECES

Start with a square piece of paper. Ordinary copy paper is fine, but paper with sides of different colors gives better results. Mark the midpoints of all sides of the square. Then connect the midpoints and vertices of the square with four lines so the square is divided into 9 parts (Sundara Row<sup>1</sup> 1905).

### Construction<sup>2</sup>



### Questions for the curious

- What polygons result from the division of the square?<sup>3</sup>
- What part of a large square (sheet) is the quadrilateral in the center?
- What is the distance from a vertex of the center quadrilateral to the vertices of the large square?
- Is the resulting pattern the only solution that divides a square into 9 parts according to the conditions specified in the description of the construction?

<sup>1</sup> Row Sundara T., 1966, Geometric Exercises in Paper Folding, Dover Publications, Nowy Jork (reprint of 1905 edition)

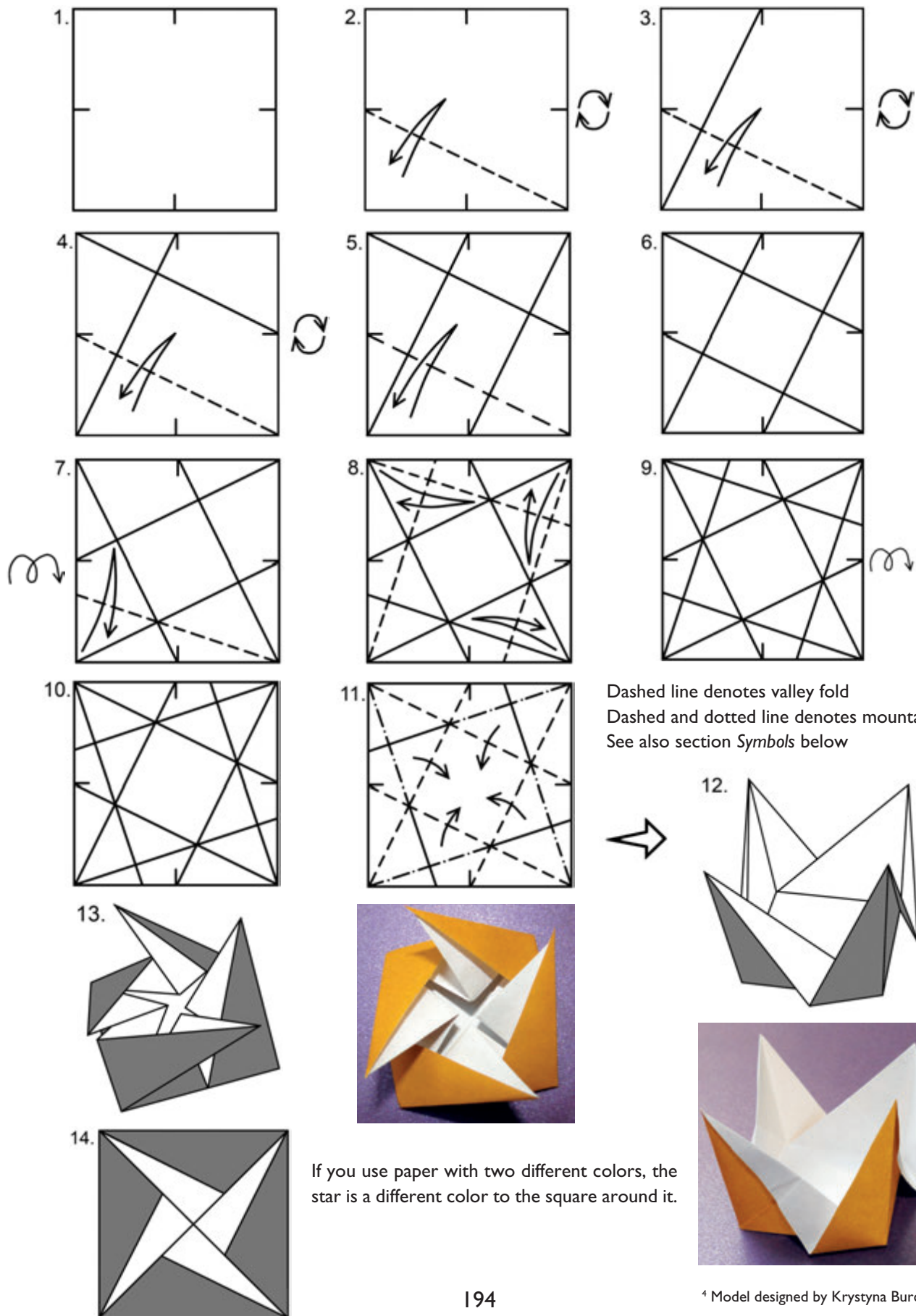
<sup>2</sup> Observe how the vertices of the square move and analyze their new positions. It is a valuable experience during assembly

<sup>3</sup> You can answer the question concerning the nature of the polygon in the middle of the square, after careful observation of the types of polygons constructed, their symmetrical arrangement relative to each other and their interior angles. These properties are easier to observe when you arrange two folded sheets side by side.



### 3 A STAR IN A SQUARE

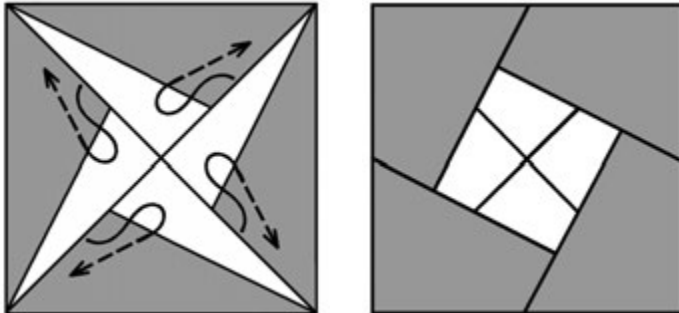
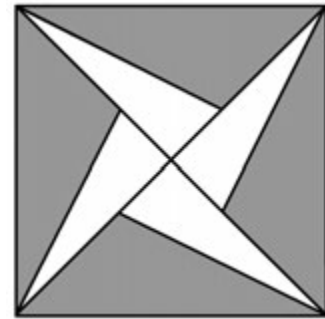
First fold lines on a square sheet of paper as shown below (step 1-10). Observe rotation signs and turn-paper-over signs (for explanation of the signs see the end of this chapter). Then form a star<sup>4</sup> (different lines show different folding directions).



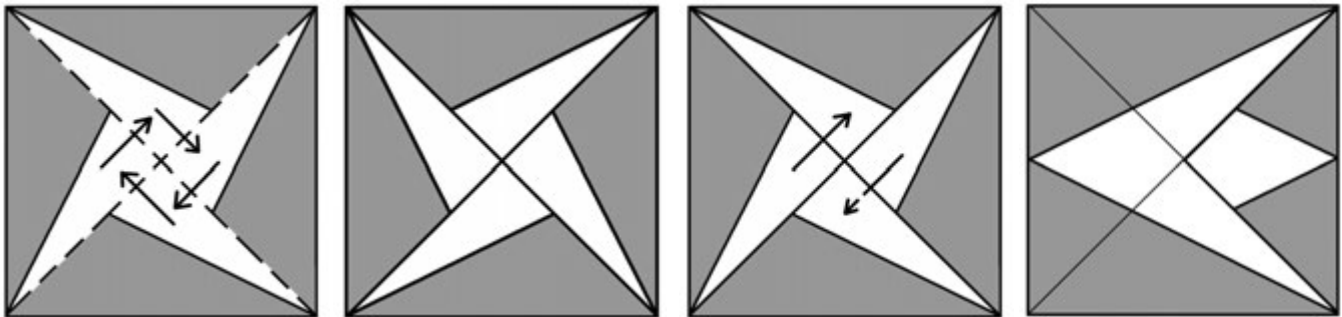


### Questions for curious

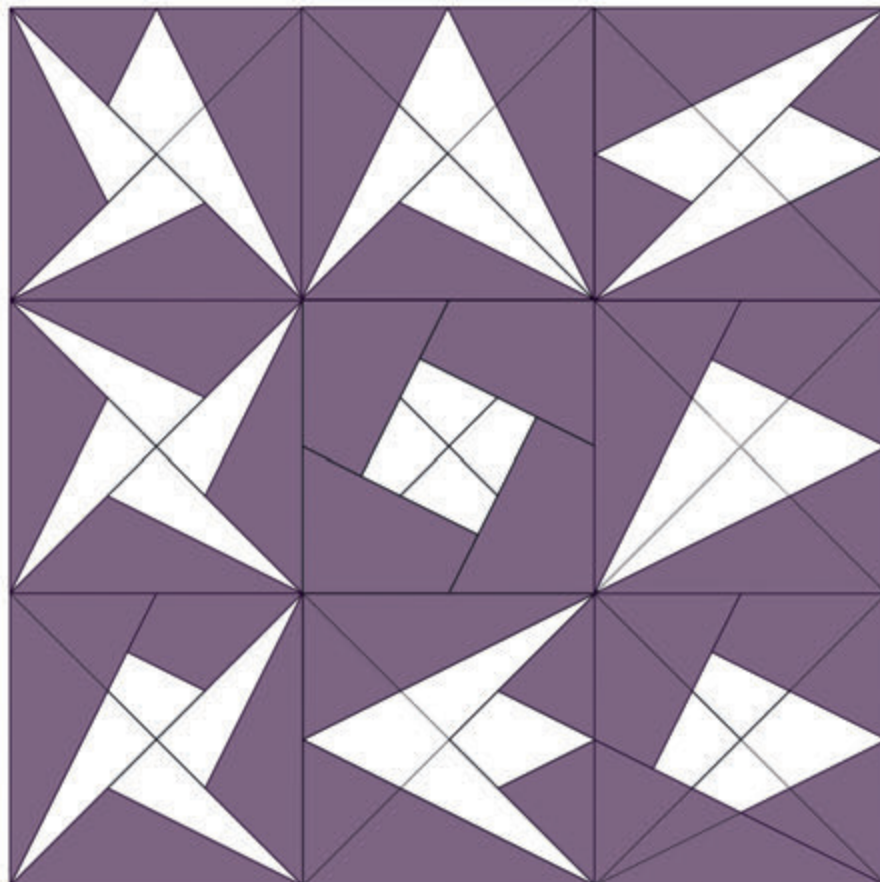
- What part of the square is the star (what is the area of the star compared to the square tile)?
- Tuck the points of the star into the pockets behind them. What part of the tile is the small square inside (the square consists of four quadrilaterals)?



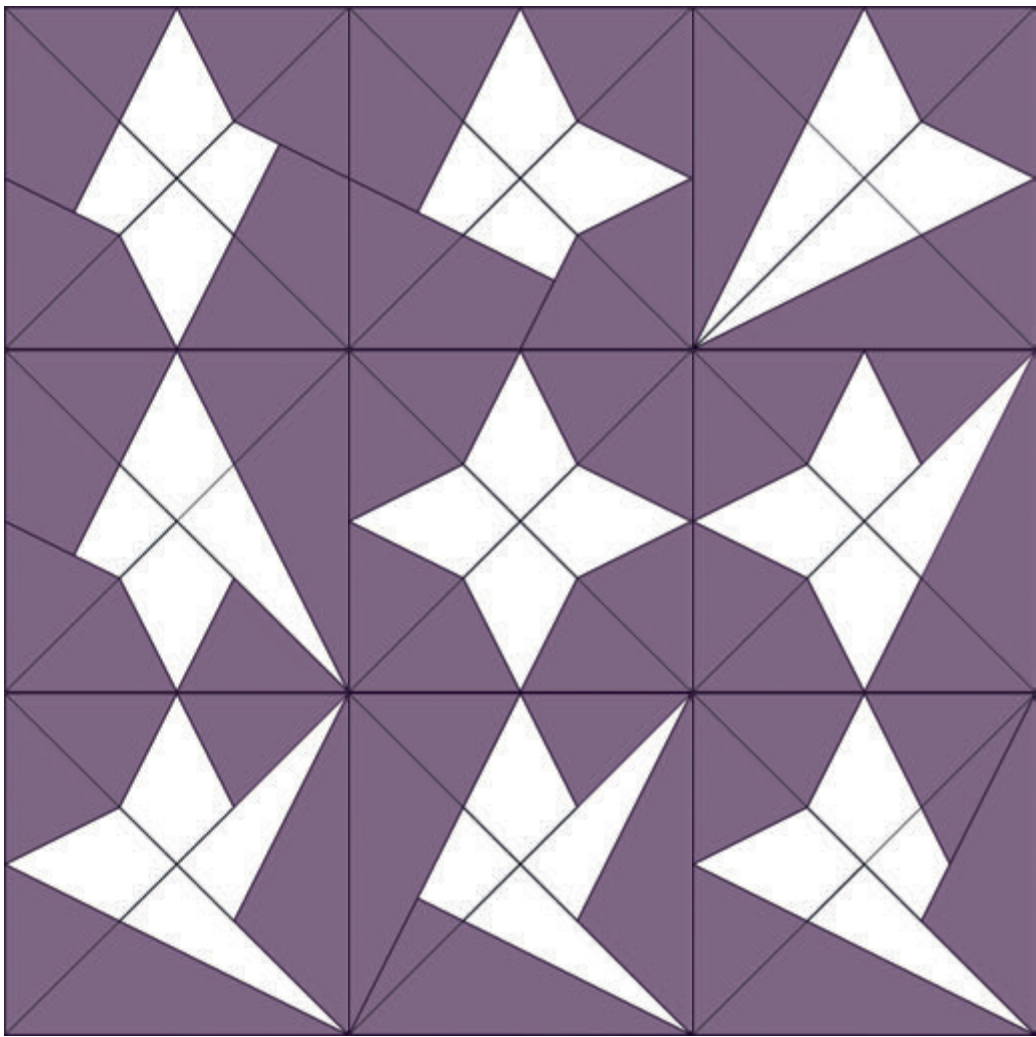
- It is possible to rearrange the flaps of the star in several ways: flap them, flap behind other flap or tuck into a pocket. What patterns on a tile can you create (without making any additional fold lines)?



- Make tiles with the following patterns (without making additional fold lines)



- Make tiles with the following patterns (without making additional fold lines)



## 4 DRAGON EGG

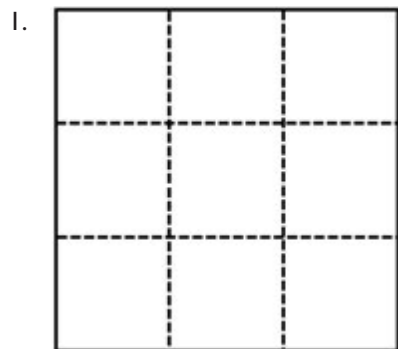
Dragon Egg is a simple model designed by Krystyna Burczyk in 2009. It is very efficient – a reasonable size of paper results in quite a large final form.

You need 12 square pieces of paper, standard 80 g/m<sup>2</sup> color copy paper is fine. The best size of paper is 10 to 15 cm.

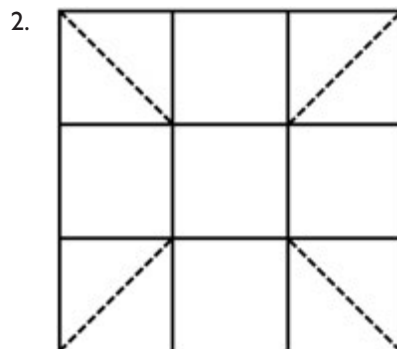


### A module

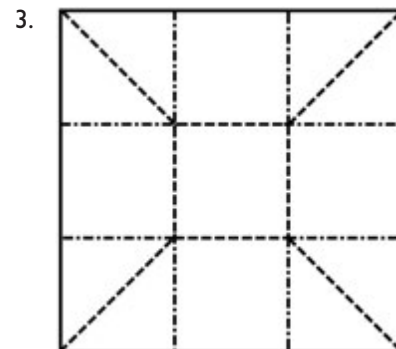
#### Questions for curious



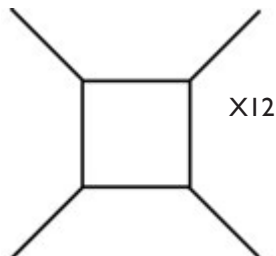
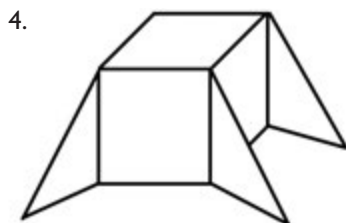
Divide the square into 9 squares (divide each side of the square paper paper into thirds).



Fold 4 small diagonals.

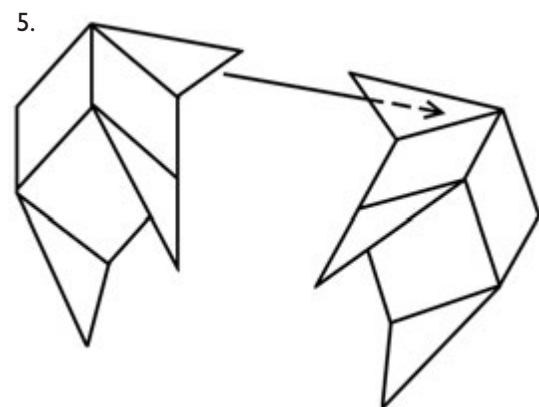


Fold lines as shown. This is the view from underneath.

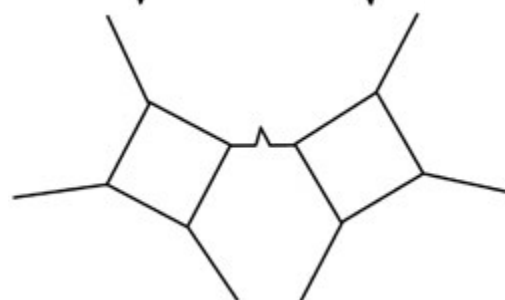
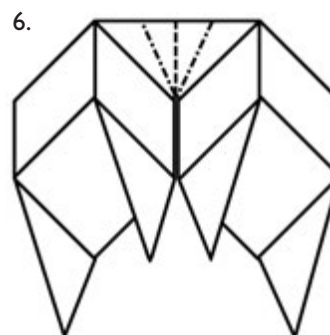


Completed module. Make 12 of them.

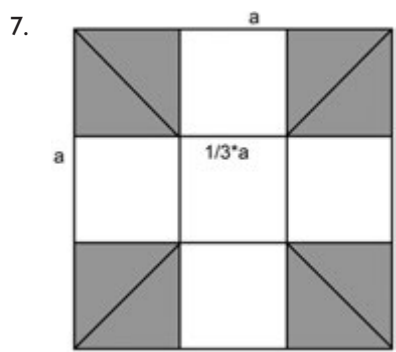
### Joining modules



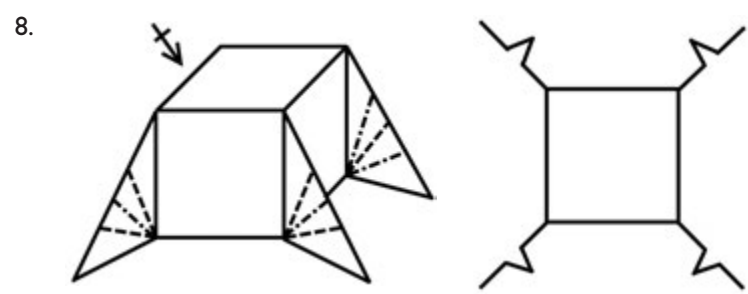
Slide a flap of the first module into a pocket (inside a flap) of the second module. Slide in as far as possible.



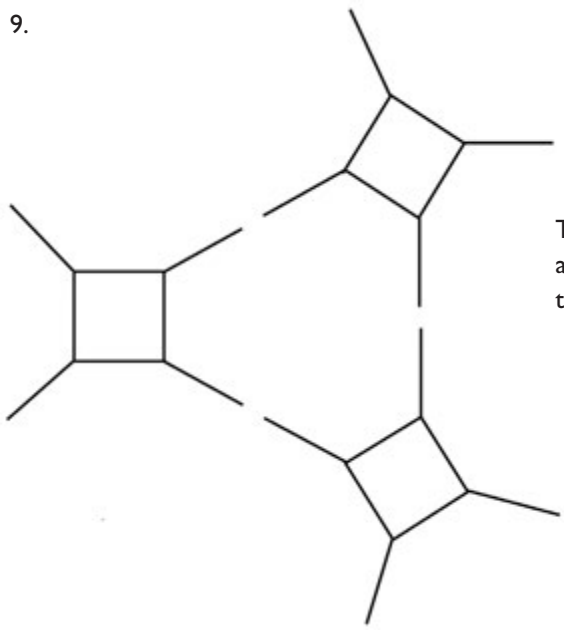
Fold a zigzag with radial lines.



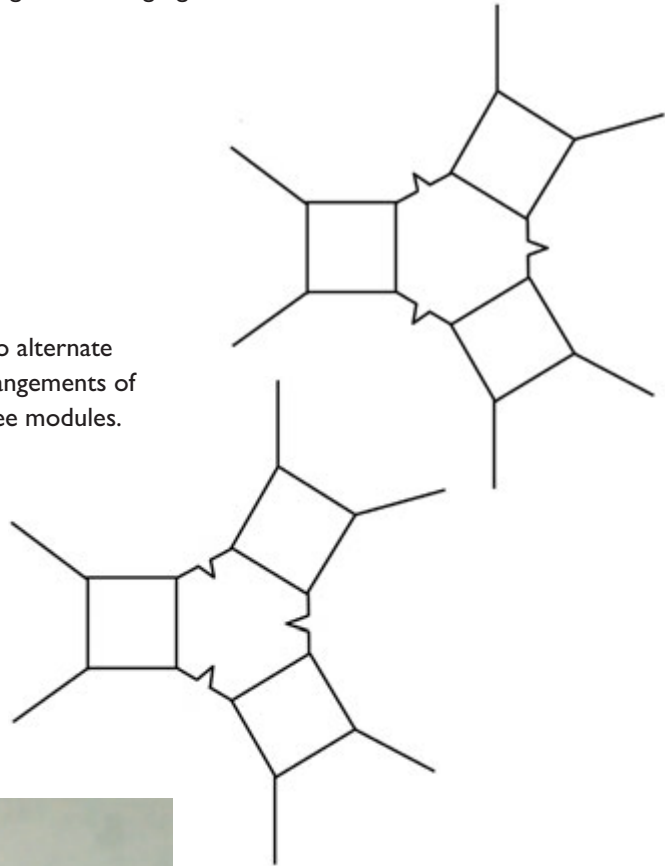
Gray area marks zigzag part of a module.



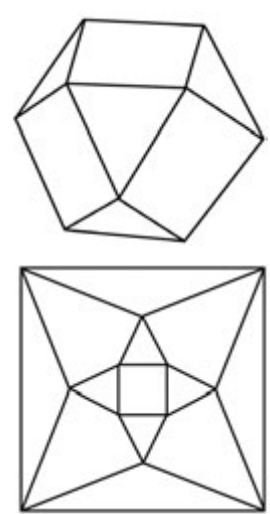
Arrangement of zigzags in a module.



Two alternate arrangements of three modules.



Assemble modules according to the cuboctahedron structure (modules correspond to vertices). There are four triangles around each square and three squares around each triangle.

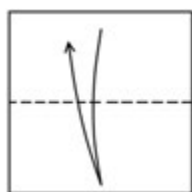






- Is division into thirds (step 1) essential for this construction? If not, what conditions guarantee that the folding sequence works?
- Is the paper's square shape essential for this construction? If not, what shape can you use and what conditions on folding guarantee that the folding sequence works?
- Is "slide in as far as possible" in step 5 essential for this construction? If not, what variation can you make?
- Is cuboctahedron the only structure that works for the assembly process? (Hint: examine Platonic and Archimedean solids!)

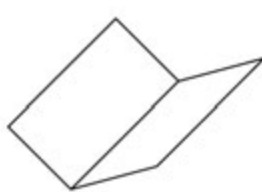
## 5 SYMBOLS



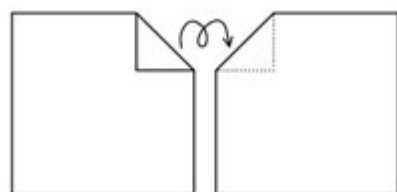
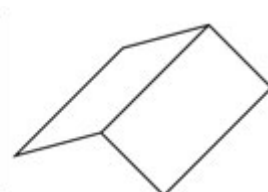
Fold and unfold



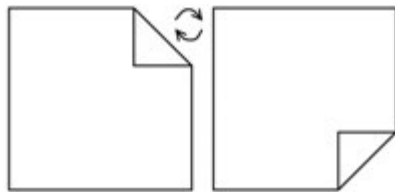
Valley fold



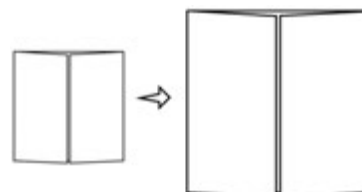
Mountain fold



Turn paper over



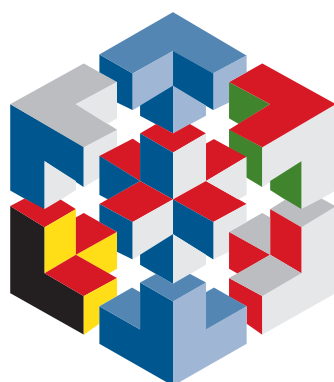
Rotate



Enlarged view



Repeat here



# VISUALITY & MATHEMATICS

EXPERIENTIAL EDUCATION  
OF MATHEMATICS THROUGH  
VISUAL ARTS, SCIENCES  
AND PLAYFUL ACTIVITIES

