THREE THEORIES AND A TEACHING MODEL

Tools for better teaching

*Timo Tossavainen**

School of Applied Educational Science and Teacher Education,
University of Eastern Finland

timo.tossavainen@uef.fi

* This material has partly been adopted from *Variation Theory – Learning From Experiencing Phenomena,* a slide show by unknown authors to be found in Internet.

1. The APOS Theory

- Learning in mathematics happen in a sequential process
- **A**ction, e.g., $Dx^2 = 2x$
- **P**rocess, e.g., A derivative is the limit of a difference quotient
- **O**bject, e.g.,Two derivatives can be added, multiplied etc.
- **S**chema, e.g., A derivative can be referred to using many different notations, it is related to continuity, integration, antiderivative, differential equations, change, higher derivatives etc.
- Dubinsky & McDonald: A Constructivist Theory of Learning in Undergraduate Mathematics Education Research

  http://www.math.kent.edu/~edd/ICMIPaper.pdf

2. Concept Definition & Concept Image

- We have and rely on both formal and informal knowledge about mathematical concepts
- Concept definition = a form of words used to specify that concept
- Concept image = the total cognitive structure that is associated with the concept, which includes all the mental pictures and associated properties and processes
- A challenge for communication: a teacher or book speaks about concept definition and a learner thinks about concept image
- Tall & Vinner: Concept image and concept definition in mathematics, with special reference to limits and continuity, *Educational Studies in Mathematics*, 12 151-169.

3. Variation Theory – The Theory of Learning

- A methodological application of phenomenography introduced by Marton
- The fundamental idea: The limits of our working memory prohibit us to pay attention simultaneously to every relevant aspects and details => there are genuinely different ways of seeing the same phenomenon
- An effective way of seeing the phenomenon can be developed if the learner can experience important variations of that phenomenon.
- To learn about something implies that you must discern the essential feature of the object of learning from the background. If there is no variation, then there is no discernment because you do not attend to things that are always the same.

What it is to vary?

- Variation assumes both **same** and **difference** - through variation the different aspects can be discerned or separated.
- To experience something includes the perception of

  - how it is like: its internal elements and configuration (the structural aspects)

  - what it is: how the thing relates to other things or certain 'external horizon' (referential aspects)

What it is to discern?

- Discernment **from and in relation to** the context means that

  - variations should be done in context so that the whole is not lost

  - variations should be also done between different contexts in order to raise up joining elements and common properties

Complex learning requires the **simultaneous awareness** of several important aspects and fluent switching between the different foci.

How to promote different ways of seeing?

- Phenomenological challenge to teaching is that the teacher and the students are seeing the object of learning from different perspectives and in different terms. It is often too easy for the teacher to take things for granted.

By creating situations that allow different approaches in doing or seeing the same thing, for example, by applying...

4. The Learning Study Model

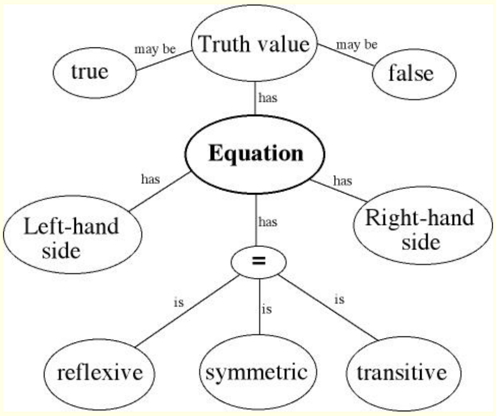
- A Mixture of design experiment (Brown, 1992; Collins, 1992)and Japanese Lesson Study approach (Marton & Pang, 2006)
- The model consists of the following consecutive steps:

1. Choosing the object of learning
2. Ascertaining students´ pre-knowledge (Pre-test)
3. Planning and implementing the lesson, video-recording
4. Post-test
5. The evaluation, revision and analysis of the lesson and the tests
6. Repeating the steps 2–5 for another group, at least, a few times.

Example: Teaching the equation concept

- Let us first consider whether this concept is so trivial that there is nothing to be taught?
- This includes asking: do we ourselves know what we really mean with the term 'equation' and is there a group of specialists who really know this concept and are unanimous about it?
- What is a suitable space of learning, i.e., the dimensions for reasonable variations?
- What do *discernment*, *simultaneity* and *variation* mean in the case of the equation concept?

- For example, $11 - 3 \neq 5$ shows that the concept is quite non-trivial, even experts may have difficulties with this expression.
- the suitable space of learning, i.e., the dimensions for reasonable variations could now be found by deconstructing the equation concept into more elementary sub-concepts and relations which construct the equation concept.



- Variations can be derived by introducing examples of equation-like expressions where some of these properties of equations are valid and some are not.
- Discernment is now supported by comparing different expressions and paying attention to them simultaneously.

Some useful examples and non-examples of equations

1. $x = x + 1$ (is equation with a variable, truth value = false)

2. $3 + 5 = 8$ (is equation, although has no variable, truth value = true)

3. $2x + 3 - 1 - x = 2x - x + 3 - 1 = x + 2$

(is not equation because of the violation of syntax)

4. $3x \approx 2$

5. $11 - 3 \neq 5$

6. $2x \leq 5x$ (4.–6. are not equations because of the involved relation)

Experiences from the real case studies

- If LSM is applied several times, the teacher gets quite objective knowledge about which aspects of the equation concept are most difficult for the learners.
- Applying the LSM usually reveals that learners have surprising and incorrect conceptions even about most simple entities.
- For example, Tossavainen, Attorps and Väisänen, On mathematics students' understanding of the equation concept, *Far East Journal of Mathematical Education* 6 (2) (2011), 127–147 revealed that misconceptions related to the syntax and the truth value of equations are very common. What could explain this phenomenon?

Exercise

- Form a small group of 2–3 person and choose one of the following topics:
  - 1) a bijective function (one-to-one mapping),
  - 2) a monotone function,
  - 3) the derivative of a function.
- Design a plan for a lesson on your topic and introduce it to another small group.
- On the basis of the feedback, discuss how you could improve your plan for the next lesson on the same topic.

What kind of obstacles and challenges can occur when the VT and LSM are applied?

- The procedural aspects dominate students' thinking and learning.
- To investigate concept requires conceptual and structural analysis.
- Discernment may require radical variations. Many concepts do no allow that. For example, seeing variation in the truth value of equation-like expression is easily shadowed by the symbolic notation, which, however cannot be avoided.
- The deep conceptual understanding is often beyond the available representations of the concept to be studied.
- Varying between the different representations and examples of conceptual knowledge easily loads very heavily the working memory of the learner. However, applying VT and LSM has given encouraging results also in advanced mathematics teaching, e.g., Attorps, Björk, Radic & Tossavainen, Varied ways to teach the definite integral concept. TO APPEAR IN *International Electronic Journal of Mathematics Education* (2014).