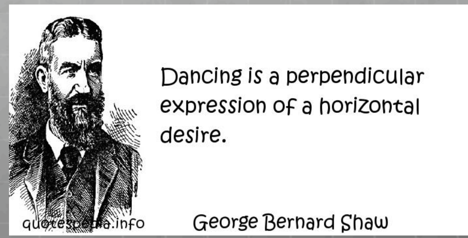
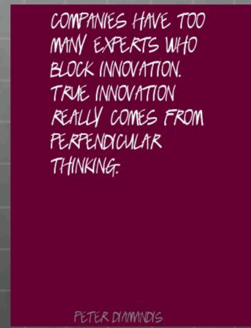
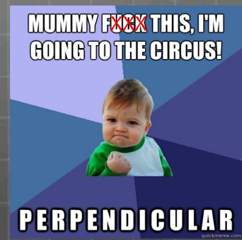
Timo Tossavainen, Eger, July 2013

# On perpendicularity

– and a few words about parallelism, too




http://www.youtube.com/watch?v=vnnwfcDcNIY

## Teaching perpendicularity and parallelism

- Are perpendicularity and parallelism so trivial or familiar concepts from the real life that every student learns the essential facts about them on her/his own?
- Is is possible to teach proper axiomatic thinking for senior high school students?
- The conceptual and procedural knowledge of mathematics: Should we first understand in order to be able to do mathematics or is it vice versa?

## Properties of perpendicularity

- Symmetry: if $a \perp b$, then also $b \perp a$
- Irreflexivity: for none of elements, $a \perp a$
- Transitivity?
- Some other property?

$$a \perp b \perp c \perp d \Rightarrow a \perp d \ ?$$

## Three alternatives

- A binary relation is perpendicularity if it is symmetric and irreflexive.
- Perpendicularity is not a binary relation but, for example, a trinary relation.
- There is not a universal perpendicularity but several different perpendicularities in different contexts.

## An axiom system for (planar) perpendicularity and parallelism

A1: $\forall a : \neg\, a \perp a$

A2: $\forall a,b : a \perp b \Rightarrow b \perp a$

A3: $\forall a,b,c,d : a \perp b \perp c \perp d \Rightarrow a \perp d$

A4: $\forall a : \exists b : a \perp b$

A5: $\forall a,b : a \parallel b \Rightarrow \exists c : a \perp c \perp b$

A6: $\forall a,b,c : a \perp b \perp c \Rightarrow a \parallel c$

## Some results

**Theorem 1.** Parallelism || is an equivalence relation.

**Theorem 2.** $\forall a,b : a \parallel b \Rightarrow \neg\, a \perp b.$

**Theorem 3.** $\forall a,b,c : a \parallel b \wedge b \perp c \Rightarrow a \perp c.$

These results and many other verifiable propositions in this axiom system are compatible with the model of Euclidean geometry in plane.

## Another model

**Example 1.** Let $X = \{0,1\}$ and

| $\perp$ | 0 | 1 |
|---|---|---|
| 0 | no | yes |
| 1 | yes | no |

| $\parallel$ | 0 | 1 |
|---|---|---|
| 0 | yes | no |
| 1 | no | yes |

## More models

**Example 2.** $X = R\setminus\{0\},\ x \perp y \Leftrightarrow xy < 0,\ x \parallel y \Leftrightarrow xy > 0.$

**Example 3.** $X = R\setminus\{-1,0,1\},\ x \perp y \Leftrightarrow |xy| = 1,\ x \parallel y \Leftrightarrow |x| = |y|.$

**Example 4.** In the set of all lines in the Euclidean plane, define that two lines are perpendicular if the smallest angle between them measures 45°, and parallel if they are parallel in the ordinary sense or the angle between them measures 90°.

## Another axiom system for (algebraic) perpendicularity

A1: $\forall a \neq 0 : \neg\ a \perp a$

A2: $\forall a,b : a \perp b \Rightarrow b \perp a$

A3: $\forall a : \exists b : a \perp b$

A4: $\forall a,b : a \perp b \Rightarrow a \perp -b$

A5: $\forall a,b,c : a \perp b \wedge a \perp c \Rightarrow a \perp (b+c)$

## Some facts about algebraic perpendicularity

- It is compatible with every vector space, the axioms are derived from the property that the inner product for two perpendicular vectors is zero.
- In algebraic context, interesting questions about perpendicularity are different from those in geometric context.
- An example of interesting perpendicularity: in the set of integers, ***a* and *b* are perpendicular if and only if they are relatively prime.**

## Pedagogical conclusions

- Axiomatic approach helped us to find new aspects and even new results even on very old concepts.
- We extended our conceptual understanding about perpendicularity and parallelism through a procedural approach.
- On the other hand, the operationalization of these concepts required that we already have an internalized view of the domain of possible axioms and, in general, suitable criteria for choosing proper axioms etc.

# References

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3. P. Haukkanen, M. Mattila, J. K. Merikoski & T. Tossavainen (2013). Perpendicularity in an Abelian group. *International Journal of Mathematics and Mathematical Sciences, Volume 2013, Article ID 983607.*